TTMF parte 2

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PART 2

Eigenvalues and eigenvectors of Matrices

Matrices play an important **role in economics**: for instance the input-output matrix, the Markov matrix, the Hessian matrix related to optimization problems, the Jacobian matrix used in difference and differencial equations.

DEF MATRIX: a Matrix is a rectangular array of numbers, hence any **table containing real data** is a matrix. The numbers are organized in row and columns, and we consider a matrix having m rows and n columns, called **(mxn) matrix**. Each entry of the matrix is an **element** belonging to row i and column j, so it is indicated as a_{ij}.

EX 1: *consider the following matrices*

$$A=\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}; B=\begin{pmatrix} \sqrt{2} & \ln 3 \\ 5 & 0 \\ 0 & 3 \end{pmatrix}; C=\begin{pmatrix} 10 & 6 & 9 & 0 \\ -4 & 8 & 1 & 0 \end{pmatrix}$$

Then A is a (2x2) matrix, B is a (3X2) matrix while C is a (2X4) matrix.

The element of A belonging to the first row and second column is 0, that is $a_{12}=0$.

While for matrix B we have $b_{32}=3$, $b_{21}=5$.

Among the set of all matrices we focus on particular kind of matrices.

DEF. SQUARE MATRIX: a Matrix is called **square matrix** if m=n, i.e. the number of rows and columns is equal. Hence we can talk about a square matrix of order n.

EX 7: *consider the following matrices,*

$$A=\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}, B=\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 0 & 1 & 0 \end{pmatrix}, C=\begin{pmatrix} 7 & 5 \\ -1 & 3 \\ -2 & -8 \end{pmatrix}$$

Then A is quare of order 2, B is square of order 3, C is not square.

Consider a square matrix, then the following definitions can be given.

DEF. MAIN DIAGONAL OF A SQUARE MATRIX: the main diagonal of a square matrix is given by the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & a_{3,3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n,1} & a_{n,2} & \cdot & \cdot & a_{n,n} \end{pmatrix}$$

DEF. SYMMETRIC MATRIX: A square matrix is said to be symmetric if it does not change with transposition (that is changing rows with columns). In this case the elements that are in symmetric position with respect to the main diagonal are equal.

DEF. TRIANGULAR MATRIX: a triangular matrix is a matrix in which the elements above or below the main dagonal are zero.

DEF. DIAGONAL MATRIX: Is a matrix in which the elements that do not belong to the main diagonal are zero.

DEF. IDENTITY MATRIX: Is a matrix in which the elements that do not belong to the main diagonal are zero while all the elements on the main diagonal are 1.

EX 8:

$$A=\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{pmatrix}, B=\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}, C=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, I=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A is symmetric, B is triangular, C is diagonal, I is the identity matrix of order 3

Consider a square matrix A(nxn) and a column vector x̲ of dimension n. Then the product Ax̲ gives a column vector y̲ of dimension n. Usually a vector change direction when it is multiplied by A.

We recall that, if A is (nxn) and $\underline{x}$ is a column vector with n elements, then it is possible to compute **A$\underline{x}$** as follows:

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},\ \underline{x}=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \Rightarrow A\underline{x}=\begin{pmatrix} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n \\ \dots \\ a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n \end{pmatrix}$$

EX 9: *consider the following matrices and vectors:*

$$A=\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix},\ B=\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\underline{x}=\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \underline{y}=A\underline{x}=\begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

in this case $\underline{x}$ and $\underline{y}$ are proportional, and $\underline{y}=3\underline{x}$

$$\underline{x}=\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \underline{y}=B\underline{x}=\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

in this case $\underline{x}$ and $\underline{y}$ do not have the same direction

Work with matrices

With MatLab it is possible to calculate the **product between a matrix and a vector,** the operator to be used is *****

EX 10: *save the following matrix and vectors, calculate the products between the Matrix and the vectors, and verify if a vector having the same direction of the initial one is obtained.*

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \underline{x}1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \underline{x}2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \underline{x}3 = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

…Work with matrices

…EX10: *save the following matrix and vectors, calculate the products between the Matrix and the vectors, and verify if a vactors havning the same direction is obtained.*

```
>> A=[1 0 2; -1, 4, 0;1,0,1]
>> x1=[0 2 0]'
>> x2=[-1;1;0]
>> x3=[3,1,2,4]'
```

```
>> A*x1
   0
   8
   0
>> A*x2
  -1
   5
  -1
>> A*x3
Error using  *
Inner matrix dimensions must agree.
```

A*x1=4x1 so the obtained vector has the same direction of the initial one;

A*x2 is not proportional to x2;

A*x3 cannot be computed as the product is not defined

PROBLEM:
Consider a square matrix A(nxn), then we want to know **if there exists a vector x(nx1) such that the product Ax will give a vector having the same direction of x.**

DEF. EIGENVALUE AND EIGENVECTOR: let A be a square matrix, then if there exists a vector x and a real number λ such that $A\underline{x}=\lambda\underline{x}$ then λ is called **eigenvalue** while x is called **eigenvector.**

This problem can be formalized as follow:

Determine x, if it exists, such that $A\underline{x}=\lambda\underline{x}$, where λ is a real number.

Notice that solving $A\underline{x}=\lambda\underline{x}$ is equivalent to $A\underline{x}-\lambda I\underline{x}=\underline{0}$ that is

$$(A-\lambda I)\underline{x}=\underline{0}, \quad (*)$$

Where I is the identity matrix.

Equation (*) is a homogeneous system hence it always admits the null solution, that is $\underline{x}=\underline{0}$ is a solution of equation (*) for all λ. This is a trivial solution.

From basic linear algebra:

The system

$$(A-\lambda I)\underline{x}=\underline{0}, \quad (*)$$

admits a non-trivial solution iff (if and only if) **$\det(A-\lambda I)=0$ (**)**

Hence the eigenvalues of A are the solutions of equation (**).

Notice that if A is (nxn) then equation (**) is a polynomial equation with degree n. The set of eigenvalues of a matrix is called **spectrum**.

DEF: $\det(A-\lambda I)$ is called **characteristic polynomial**.

We recall that, **the determinant** of a square matrix A, namely det(A) or |A|, is a number associated to A.

If A is (2x2) then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

If A is (3x3) then $$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + \\ - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

If A is of higher order the rule for calcuation must be used or MatLab can be used.

With MatLab: the command **det(A)** will find the determinant of a square matrix A.

EX 11: *calculate with MatLab the determinant of the following matrix*

$$A = \begin{pmatrix} 1 & 2 & 6 & 4 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

```
>> A=[1 2 6 4;0 2 1 0; 0 0 -1 1;0 0 0 3]
A =
    1    2    6    4
    0    2    1    0
    0    0   -1    1
    0    0    0    3
>> det(A)
ans =
   -6
```

Notice that: A is a triangular matrix so its determinant is given by the products between the elements belonging to the main diagonal!

Once obtained one eigenvalue, that is one solution of equation (**), then the corresponding **eigenvector** can be determined by solving the linear system (*).

Notice that:

1) Equation (**) always admits n roots, those are the eigenvalues of A, but they can be real numbers or complex numbers (in pair).

2) Anyway, if **A is diagonal or triangular**, then it is easy to show that the eigenvalues are real and they are given by the elements of the main diagonal.

3) It can be proved that **if A is symmetric** then it admits only real eigenvalues.

4) The solution of equation (**) can be difficult to be obtained analytically, especially if $n>2$.

EX 12: *find the eigenvalues and the eigenvectors of the following matrix.*

$$A=\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$$

$$A-\lambda I=\begin{pmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{pmatrix}\Rightarrow$$

$$\begin{vmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{vmatrix}=(1-\lambda)(4-\lambda)-4=4-\lambda-4\lambda+\lambda^2-4=0$$

$$\Rightarrow \lambda^2-5\lambda=0\Rightarrow \lambda(\lambda-5)=0\Rightarrow \lambda_1=0,\lambda_2=5$$

... hence in this case we obtain two dinstinct real eigenvalues.

We can then find the two eigenvectors.

The eigenvectors associated to the null eigenvalue can be obtained by solving:

$$(A - 0I)\underline{x} = \underline{0} \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$x_1 = -2x_2, \forall x_2 \in R$$

While the other eigenvector can be obtained by solving

$$(A - 5I)\underline{x} = \underline{0} \Rightarrow \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$x_1 = -1/2x_2, \forall x_2 \in R$$

EX 13: *find the eigenvalues of the following matrix.*

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \begin{aligned} &= (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 2] \\ &= (1-\lambda)(-\lambda)(3-\lambda). \end{aligned}$$

Thus A admits 3 real eigenvalues: 0, 1, 3.

EX 14: *to find the eigenvalue of matrix* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

The following equation must be solved $\lambda^2 + 1 = 0$

Since no real roots are admitted, in this case the two eigenvalues are complex.

In some cases it can be very difficult to find the eigenvalues and eigenvectors of a matrix analytically.

For instance, if the matrix has order greater then 2, then the roots of a polynomial of high degree must be found. Normally numerical methods must be applied in this case.

Using **MatLab** it is possible to find the eigenvalues and eigenvectors of a matrix.

Let A be a square matrix. The command

[V L]=eig(A)

will give two outputs:

-**V** is a matrix and the columns are the eigenvectors of A (notice that MatLab will show the eigenvectors having norm 1);
-**L** is a diagonal matrix and the elements of the main diagonal are the eigenvalues of A.

Notice that: the command eig(A) (that is without output specification) will give only the vector of eigenvalues of A.

EX 15: *find by using MatLab the eigenvalues of the following matrix.*

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 2 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 2 & 3 & 0 & 0 \end{pmatrix}$$

```
>> A=[1 2 0 2; 2 2 1 3; 0 1 -1 0; 2 3 0 0]
A =
     1     2     0     2
     2     2     1     3
     0     1    -1     0
     2     3     0     0
>> eig(A)
ans =
   -2.4568
   -1.2403
   -0.1679
    5.8650
```

Notice that: since A is symmetric then its eigenvalues are real numbers

EX 16: *find by using MatLab the eigenvalues and the eigenvectors of the following matrix.*

$$B = \begin{pmatrix} 6 & 12 & 19 \\ -9 & -20 & -33 \\ 4 & 9 & 15 \end{pmatrix}$$

```
>> B=[ 6    12    19; -9   -20   -33; 4     9    15 ];
>> [V,L]=eig(B)

V =
   -0.4741   -0.4082   -0.4082
    0.8127    0.8165    0.8165
   -0.3386   -0.4082   -0.4082

L =
   -1.0000         0         0
         0    1.0000         0
         0         0    1.0000
```

Notice that: there is a double eigenvalue and the second and third eigenvectors are then equal.

Homeworks 2

-Find the determinant, eigenvalues and eigenvectors associated to the following matrices (first analytically and then check with MatLab your results).

$$A=\begin{pmatrix} 1 & -1 \\ -6 & 2 \end{pmatrix}; \quad B=\begin{pmatrix} 3 & 0 \\ 2 & 3 \end{pmatrix}$$

-Find the determinant and the eigenvalues of the following matrix.

First try analytically and then check your results with MatLab.

$$C=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

…Homeworks 2

-Find with MatLab the eigenvalues of the following matrices.

$$A=\begin{pmatrix}1 & 2 & 3 & 2\\ 2 & 1 & -3 & 0\\ 0 & 0 & -1 & 0\\ 0 & 3 & 0 & 4\end{pmatrix}, B=\begin{pmatrix}1 & 1 & 5\\ 6 & 5 & 0\\ -8 & 4 & 3\end{pmatrix}$$

-Find with MatLab the eigenvalues and eigenvectors of the following matrices

$$C=\begin{pmatrix}1 & 4 & 3 & 2\\ -1 & 2 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 3 & 0 & 3\end{pmatrix}, D=\begin{pmatrix}1 & 1 & 4\\ 1 & -2 & 4\\ 4 & 4 & 3\end{pmatrix}$$

DEF. SPECTRUM: the set of eigenvalues of a matrix is called **SPECTRUM** of the matrix.

EX 17: *consider the matrix* $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ *then its spectrum is given by {-1,1}.*

DEF. QUADRATIC FORM: a quadratic form, namely Q(x̲), is an homogeous polynomial of degree 2 in the variables x_1, x_2, …, x_n.

EX 18: *consider the following polynomials*

$$Q_1 = 2x_1^2 + 3x_2^2 - x_3^2 + 4x_1x_3 - 6x_2x_3$$

$$Q_2 = -x_1^2 + x_2^2 - 2x_1x_2$$

$$Q_3 = -3x_1 + x_2^2 - 2x_1x_2$$

$$Q_4 = -x_1^2 + x_2^2 - 2x_1x_2 + 4$$

Then: Q_1 is a quadratic form with 3 variables; Q_2 is a quadratic form with 2 variables; Q_3 and Q_4 are not quadratic forms

In general a quadratic form **can be written as follows**:

$$Q(x_1, x_2, ..., x_n) = a_{11}x_1^2 + a_{22}x_2^2 + ... + a_{nn}x_n^2 + a_{12}x_1x_2 + ... + a_{n1}x_nx_1 + ...$$

where a_{ij} is the coefficient of the term x_ix_j in the polynomial.

Then the following definition can be given.

DEF. MATRIX ASSOCIATED TO A QUADRATIC FORM: any quadratic form Q(x) can be written as x'Ax where x is a column vector having n components while A is a SYMMETRIC matrix of order n. The matrix A associated to Q(x) is given by:

$$A = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & ... & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} & ... & \frac{1}{2}a_{2n} \\ ... & ... & ... & ... \\ \frac{1}{2}a_{1n} & \frac{1}{2}a_{2n} & ... & a_{nn} \end{pmatrix}$$

EX 19: *consider the quadratic form*

$$Q = 2x_1^2 + 3x_2^2 - x_3^2 + 4x_1x_3 - 6x_2x_3$$

Then the matrix A associated to the quadratic form is given by:

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{pmatrix}$$

While if we have the following matrix

$$B = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

Then the corresponding quadratic form is given by:

$$Q = -2x_1^2 + x_3^2 + 2x_1x_2 + 4x_1x_3$$

DEF. CLASSIFICATION OF QUADRATIC FORMS. We give the following definition.
A quadratic form is said to be:

Positive definite if $Q(\underline{x}) > 0, \forall \underline{x} \neq \underline{0}$

Negative definite if $Q(\underline{x}) < 0, \forall \underline{x} \neq \underline{0}$

Positive semidefinite if $Q(\underline{x}) \geq 0, \forall \underline{x} \text{ and } \exists \underline{x} \neq \underline{0} : Q(\underline{x}) = 0$

Negative semidefinite if $Q(\underline{x}) \leq 0, \forall \underline{x} \text{ and } \exists \underline{x} \neq \underline{0} : Q(\underline{x}) = 0$

Indefinite if $\exists \underline{x} : Q(\underline{x}) > 0 \text{ and } \exists \underline{y} : Q(\underline{y}) < 0$

EX 20:

the following quadratic form $Q = x_1^2 - 2x_1x_2 + x_2^2$

Is positive semidefinite. In fact:

$$Q = x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$$

$$\Rightarrow Q \geq 0 \text{ but if we consider } x_1 = 1, x_2 = 1 \text{ then } Q = 0$$

EX 21:

the following quadratic form $Q = -x_1^2 - x_2^2$

Is negative definite. In fact it cannot be positive and it is zero only if both x_1 *and* x_2 *are zero.*

The classification of a quadratic form can be difficult to be determined. But it is very easy to conclude by considering the eigenvalues of the matrix associated to the quagìdratic form.

TH. on CLASSIFICATION OF QUADRATIC FORMS. Consider a quadratic form Q and let A be the matrix associated to Q. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. Then Q is

Positive definite iff all the eigenvalues of A are positive,

Negative definite if all the eigenvalues of A are negative,

Positive semidefinite if all the eigenvalues of A are not negative and at least one is zero

Negative semidefinite if all the eigenvalues of A are not positive and at least one is zero

Indefinite if A admits both positive and negative eigenvalues

Notice that: similarly we can talk about the **definition of a matrix**.

EX 22:
Consider the following quadratic form $x_1^2 - 2x_1x_2 + x_2^2 + x_3^2$

Then the associated matrix is given by:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

With MatLab it is possible to verify that the eigenvalues are 0,1,2

And consequently the quadratic form is positive semi definite.

EX 23: *The following matrix*

$$A = \begin{pmatrix} -1 & -2 & 0 \\ -2 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$$

Is indefinite since it admits both positive and negative eigenvalues.

Homeworks 3

-Consider the following polynomial and determine if they are quadratic forms. If yes find the associated matrices.

$$Q1 = -x_2^2 + 3x_3^2 - x_1 x_3$$

$$Q2 = 2 - x_1^2 + 5x_2^2 - 2x_1 x_2$$

$$Q3 = -x_1^2 + 3x_2^2 + 2x_2 x_3 - x_1 x_3 + 5x_4^2 - 8x_2 x_4$$

$$Q4 = 4x_1 x_2 - x_1 x_3 - 6x_2 x_3$$

$$Q5 = 3x_1^3 + x_2 x_1 - x_2^2$$

-Given the following matrices determine the corresponding quadratic forms

$$A = \begin{pmatrix} 1 & -1 & 3 & 2 \\ -1 & 2 & 0 & 3 \\ 3 & 0 & -1 & 0 \\ 2 & 3 & 0 & 3 \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 4 \\ 0 & 4 & 3 \end{pmatrix}$$

…Homeworks 3…

-Give and example of (1) a quadratic form with 3 variables that is positive definite and give an example of (2) a quadratic form with 2 variables that is indefinite.

-Given the following matrices determine the corresponding quadratic forms and their definition:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}, B = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}$$

-Determine analytically the definition of the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

…Homeworks 3

-Determine the definition of the following matrices with MatLab:

$$C=\begin{pmatrix} 1 & 4 & -1 \\ 4 & 5 & -2 \\ -1 & -2 & 6 \end{pmatrix}, D=\begin{pmatrix} -3 & -7 & 0 \\ -7 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

-Determine analytically the definition of the following quadratic forms:

$$Q_1 = 2x_1^2 - 5x_2^2 + 3x_3^2,$$

$$Q_2 = x_1^2 + x_2^2 - 4x_1x_2$$

-Determine the definition of the following quadratic forms with MatLab:

$$Q_3 = 2x_1^2 - 5x_2^2 + 3x_3^2 - 4x_1x_3 + 2x_1x_2,$$

$$Q_4 = x_1^2 + x_2^2 - 4x_1x_2 + 2x_3x_4 + 5x_4^2 - x_1x_3$$