TMMF 6

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Part 6 – MATLAB ottimizzazione libera

UNCONSTRAINED LOCAL MAX AND MIN points

Consider the function $z = f(\underline{x})$ where $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$

$f \in C^2$ is called OBJECTIVE FUNCTION

Local maximum and minimum points can be determined by MatLab while using the

OPTIMIZATION TOOLBOX

We distinguish between two problems.

A. Find local max and min pt of a **function of two real variables**
B. Find local max and min pt of a **function of more then two real variables**

The first case:

A. Find local max and min pts of a **function of two real variables**

PRELIMINARY GRAPHICAL ANALYSIS

In this case the **graph** and the **level curves** of the function can be plotted, hence a preliminary graphical analysis can be done. It is useful to understand:

- If the function admits local max or min
- The approximative location of the candidate points

EX1 Find the local max and min of the following function $z = x^2 y^2 + 2x^2 y + y^2 + 6y$

PRELIMINARY ANALYSIS

It is useful to plot the graph and the level curves

```
>> z=@(x,y) x.^2.*y.^2+2*x.^2.*y+y.^2+6*y;
>> ezcontourf(z,[-10 10], [-10 10])
```

and for the graph

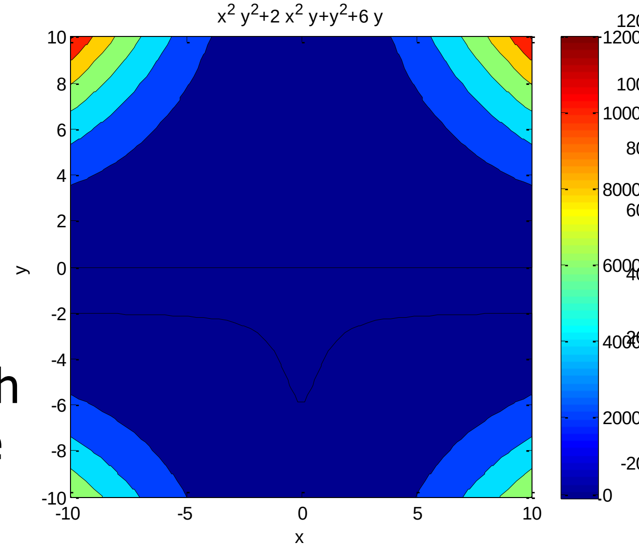
```
>> ezmesh(z,[-10 10], [-10 10])
```

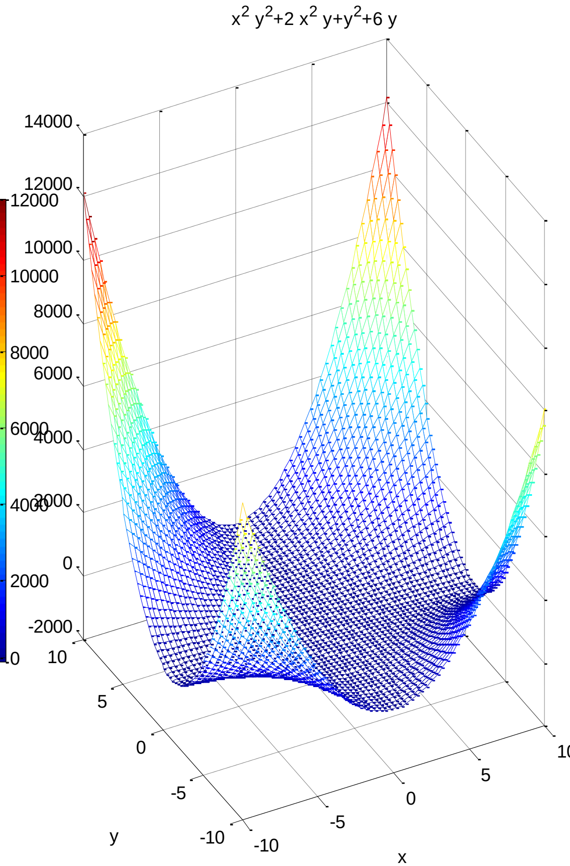
The following figure is obtained

EX1

In the **dark blue region** points (x,y) associated to "very low" z-values are obtained

A minimum point is expected to exist inside such a region, its coordinates are expected to be close to the point (0,2)





Form the preliminary analysis:

- The function admits a local minimum
- The approximative location of the local minimum pt is (0,2)

Use the optimization function

The solver that can be used to find local minimum points of a function is

fminsearch (unconstrained nonlinear minimization)

(1) First of all it is very important to underline that the algorithm is only able to find LOCAL MINIMUM pts

Hence if function f has a local maximum pt it is necessary to use the solver to find the local min of function –f, and the correspondent value of z at the maximum point is given by the opposite of the value that –f assumes at the minimum point

(2) To DEFINE THE OBJECTIVE FUNCTION:

- use the function handle @(x) where x is a vector;

-define the function by using the variables x(1) and x(2);

-don't use spaces and don't use the punctual operators.

For instance: $z=x^2+\ln(y^3+xy)$ must be defined as **@(x)x(1)^2+log(x(2)^3+x(1)*x(2))**

(3) The START POINT must be given as [x0 y0] where x0 is the initial value of variable x(1) while y0 is the initial value of variable x(2).

The function to be used is [x,fval] = fminsearch(**fun,x0**)

EX1

The objective function is:

fun=@(x)x(1)^2*x(2)^2+2*x(1)^2*x(2)+x(2)^2+6*x(2)

The start point is:

[0 -2]

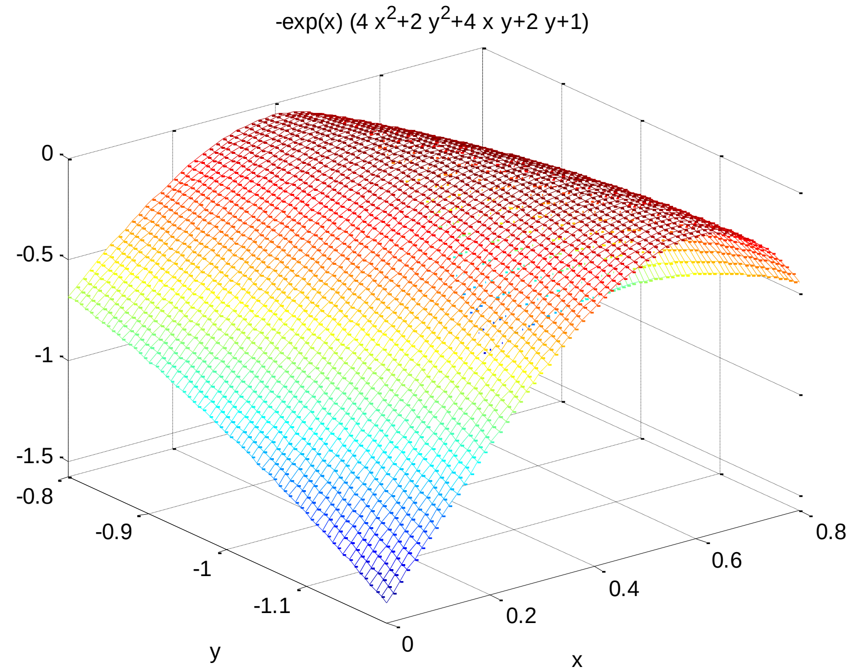
```
>> clear
>> fun=@(x)x(1)^2*x(2)^2+2*x(1)^2*x(2)+x(2)^2+6*x(2);

>> x0=[0 -2];
>> [x,fval] = fminsearch(fun,x0)

x =

   -0.0000   -3.0000


fval =

   -9.0000

>>
```

The function $z = x^2y^2 + 2x^2y + y^2 + 6y$ **has a local minimum in point (0,-3) and the correspondent value of z is -9**

EX2 Find the local max and min pts of the following function

$$f(x,y) = -e^x(4x^2 + 2y^2 + 4xy + 2y + 1)$$

The graph is the following and it seems to have a local Maximum pt

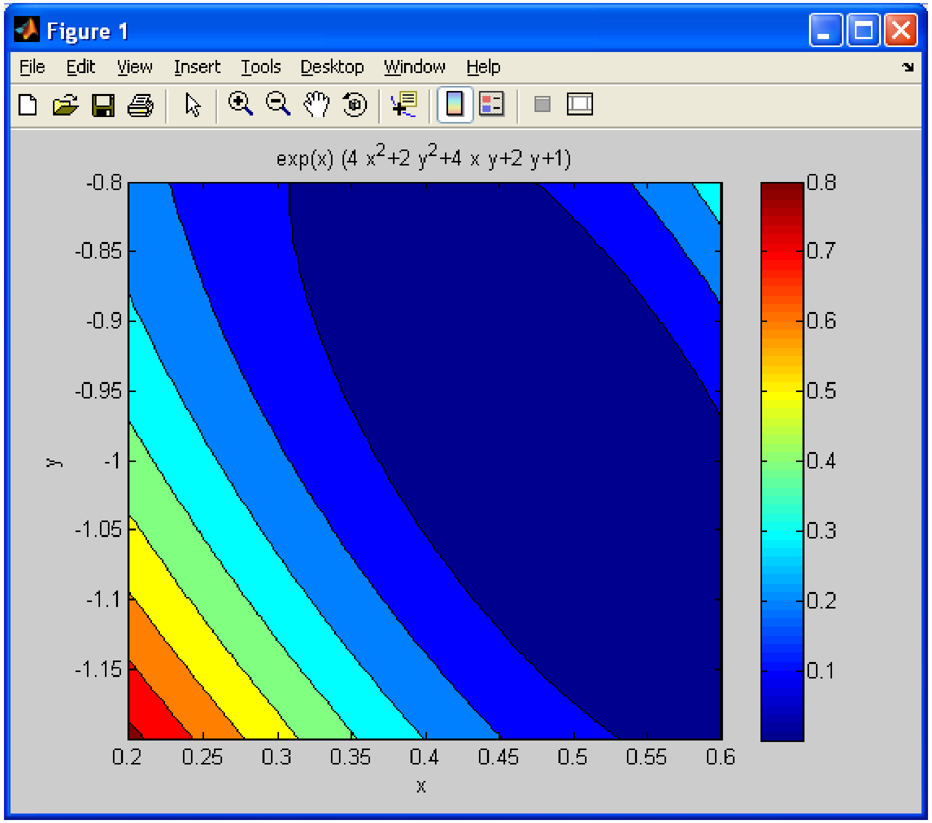


Hence we have to find the Local minimum of function

$$-f(x,y) = e^x(4x^2 + 2y^2 + 4xy + 2y + 1)$$

EX2

The level curves of function –f are depicted



We can choose the following initial vector [0.6 -0.6]

EX2

- **Define function –f (as we are interested in the max point of f)**

fun=@(x)exp(x(1))*(4*x(1)^2+2*x(2)^2+4*x(1)*x(2)+2*x(2)+1)

- **Define the start point: x0=**[0.6 -0.6]

```
>> fun=@(x)exp(x(1))*(4*x(1)^2+2*x(2)^2+4*x(1)*x(2)+2*x(2)+1);

>> x0=[-0.6 0.6]

x0 =

   -0.6000    0.6000

>> [x,fval] = fminsearch(fun,x0)

x =

    0.5000   -1.0000


fval =

   5.3767e-09
```

The local max point is (0.5,-1) and the value of z is 0

HOMEWORKS

EX 1.1

Determine local maximum and minimum point of the following functions

$$1)\ z=(2x^2+y)(x-y)$$

$$2)\ z=y^3+3y^2+x^3-3x$$

$$3)\ z=2x^2+y^2-4x$$

$$4)\ z=\frac{1}{3}x^3+x^2-3x+y^2$$

$$\left[1)M\left(-\frac{1}{4},-\frac{3}{16}\right)\ \text{2)M(-1,-2),m(1,0) 3)m(1,0) 4)m(1,0)}\right]$$

HOMEWORKS

EX 1.2

Determine local maximum and minimum points of the following functions

$$1)\ z = e^{-x^2-(y+3)^2}$$

$$2)\ z = x^4y^2 - 4x^3 - 3xy + 2y^4$$

$$3)\ z = \ln((x+2)^4 + (y-3)^2 + 4x^2 + 5)$$

The second case:

B. Find local max and min points of a **function of more then two real variables**

Notice that in this case **the preliminary graphical analysis cannot be done**.

How to determine if the function admits a local max or min?

How to determine the start point?

The answers are not easy in general. Some suggestions can be obtained while considering the critical points (to understand their number and localization).

EX3 Find the local max and min points of the following function

$$y = (x_1 - 4)^4 + e^{-x_2 x_3^2} + x_2^2 + 4x_2$$

Function f has 3 variables, it is defined on R^3
The local optimum must be found within the critical points. We try to obtain the critical points. **In such a case the critical points can be otained!**

$$\begin{cases} 4(x_1 - 4)^3 = 0 \\ -x_3^2 e^{-x_2 x_3^2} + 2x_2 + 4 = 0 \\ -2x_3 e^{-x_2 x_3^2} = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 4 \\ 2x_2 + 4 = 0 \Rightarrow x_2 = -2 \\ x_3 = 0 \end{cases}$$

Point [4 -2 0] is the unique critical point, it can be a local max or min or a saddle. We can consider this point as start point and we have to check:

- If it is a loc min of f
- If it is a local min of –f (that is a loc max of f).

If the above conditions are not satisfied then it is a saddle point!

EX3

Use **fminsearch** for function f and start point [4 -2 0]. The following answer is obtained.

Hence point [4 -2 0] is a local min and the z-value is -3.

EX4 Find the local max and min points of the following function

$$y = -\left(\frac{x_1^2}{2} + x_2^2 - 2x_2x_3 + x_3^2 \right)$$

Function f has 3 variables, it is defined on R^3

The local optimum must be found within the critical points. We try to obtain the critical points.

$$\begin{cases} -x_1 = 0 \\ -2x_2 + 2x_3 = 0 \\ 2x_2 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_3 = x_2 \\ \forall x_2 \in \mathbb{R} \end{cases}$$

In such a case there are infinite critical points of the kind (0,X,X).

We consider the Hessian Matrix given by:

$$Hf(0, X, X) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

Which is semidefinite (the eigenvalues are -4,-1,0) and consequenlty the critical points cannot be classificated. For this purpose we use the optimtool.

EX4

1. Firstly we use **fminsearch** for function f and start point [0 2 2] that is one of the critical points. The following answer is obtained.

```
>> fun=@(x)-(x(1)^2/2+x(2)^2-2*x(2)*x(3)+x(3)^2);
>> x0=[0 2 2];
>> [x,fval] = fminsearch(fun,x0)

Exiting: Maximum number of function evaluations has been exceeded
         - increase MaxFunEvals option.
         Current function value: -808617749162788251123611783881598:


x =

   1.0e+52 *

    0.0020    3.1992   -5.7931


fval =

  -8.0862e+105

>>
```

Hence **point [0 2 2] is a not a local minimum** (and the same occurs if other points are considered).

EX4

2. Secondly we use the otimtool (**fminsearch**) for function -f and start point [0 2 2] that is one of the critical points. The following answer is obtained.

```
>> fun=@(x)(x(1)^2/2+x(2)^2-2*x(2)*x(3)+x(3)^2);
>> x0=[0 2 2];
>> [x,fval] = fminsearch(fun,x0)

x =

   -0.0000    2.0001    2.0001


fval =

  -8.8818e-16

>>
```

Hence **point [0 2 2] is a local maximum of f** (and the same occurs if other points are considered, it can be verified while considering other points).

EX5 Find the local max and min points of the following function

$$y = e^{x_1} - \frac{x_1^2}{2} - 2x_1 + x_2^2 + 5x_3^2 + x_2 x_3 + x_3$$

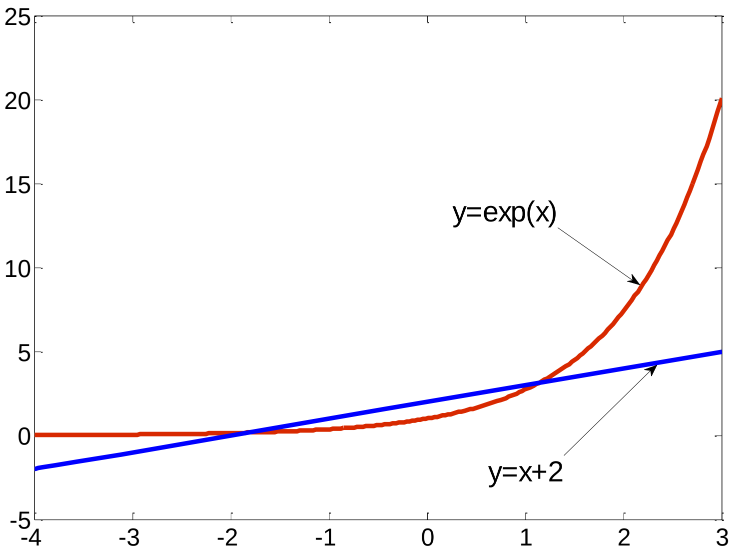
Function f has 3 variables, it is defined on R^3

The local optimum must be found within the critical points. We try to obtain the critical points.

$$\begin{cases} e^{x_1} - x_1 - 2 = 0 \\ 2x_2 + x_3 = 0 \\ 10x_3 + x_2 + 1 = 0 \end{cases} \Rightarrow \begin{cases} e^{x_1} = x_1 + 2 \\ x_3 = -2x_2 \\ -20x_2 + x_2 + 1 = 0 \end{cases} \Rightarrow \begin{cases} e^{x_1} = x_1 + 2 \\ x_3 = -2/19 \\ x_2 = 1/19 \end{cases}$$

In such a case the value of x_1 cannot be determined.

Anyway from the graphical analysis it can be observed that the first equation has two solutions: a positive solution and a negative solution.



So we consider two starting points:

A=[1 1/19 -2/19]

B=[-2 1/19 -2/19]

EX5

1. Firstly we consider the start point =[1 1/19 -2/19]. For function f we obtain the following:

```
>> fun=@(x)exp(x(1))-x(1)^2/2-2*x(1)+x(2)^2+5*x(3)^2+x(2)*x(3)+x(3);
>> x0=[1 1/19 -2/19];
>> [x,fval] = fminsearch(fun,x0)

x =

    1.1462    0.0526   -0.1053


fval =

    0.1443

>> |
```

Hence the function has a minimum point P=(1.146,0.053,-0.105)

EX5

2. Secondly we consider the start point =[-2 1/19 -2/19]. For function f we obtain the minimum point is not found.
Hence we try with function –f and the same start point and we obtain the following.

```
>> fun=@(x)exp(x(1))-x(1)^2/2-2*x(1)+x(2)^2+5*x(3)^2+x(2)*x(3)+x(3);
>> x0=[-2 1/19 -2/19];
>> [x,fval] = fminsearch(fun,x0)

Exiting: Maximum number of function evaluations has been exceeded
         - increase MaxFunEvals option.
         Current function value: -479639325110852302137939352788059538


x =

   1.0e+53 *

   -3.1036   -0.0586    0.0632


fval =

  -4.7964e+106

>> |
```

Hence the function has no minimum or maximum point close to B=(-2,1/19,-2/19)

HOMEWORKS

EX 1.3

Determine local maximum and minimum points of the following functions: find the possible critical points and conclude about them by using the optimtool

$$1)\ y = x_1^4 x_2 + 3x_3^2 - 2x_3 + 2x_2^2 - x_2$$

$$2)\ y = (x_1 - 3)^2 (x_2 - 5)^2 + e^{-x_3^2}$$

$$3)\ y = \ln((x_1 + 2)^4 + (x_2 x_3)^2 + 4x_3^2 + 5x_3)$$

HOMEWORKS

EX 1.4

Determine local maximum and minimum points and the correspondent z-values of the following functions: choose the suggested start point

$$1)\ y = 2x_1^2 + 3x_2^2 x_3^4 + 4x_2^2 + (x_1 - 1)^2, S.P.(1,1,1)$$

$$2)\ y = -\left(\ln(x_1^4 + x_1^2 + 2) + x_2^2 (x_3 - 4)^4\right), SP.(-1,-1,-1), S.P.(1,1,1)$$

$$3)\ y = -e^{-x_1^2 - 2x_2^2} + 3x_3^6 + (x_4 - 3)^2, S.P.(4,4,4,4)$$