Additional Homeworks

Mauro Maria Baldi

mauromaria.baldi@unimc.it

1 Domain of functions

Find the domain of the following functions:

1. $z = \sqrt{9 - (x^2 + y^2))} + \sqrt{y^2 - y - 2}$
2. $z = \sqrt{x^2 + y^2 - 9}$
3. $z = \sqrt{\ln(x - y)}$
4. $z = \sqrt{x^2 - 5xy + 4y^2}$
5. $z = \dfrac{x^2 - y + 5}{y^2 - 2x}$
6. $z = \ln\left(1 - x^2 - y^2\right)$
7. $z = \dfrac{y}{\cos x}$
8. $z = \sqrt{3x - 2y + 5}$
9. $z = \sqrt{\ln\left(y^2 - 2x\right)}$
10. $z = \dfrac{\sqrt{4x^2 + 9y^2 - 36}}{x^2 - 4y^2 - 4}$
11. $z = \dfrac{1}{y - x} + \sqrt{1 - xy}$
12. $z = \sqrt{4 - x^2 - y^2} + \sqrt{x^2 + y^2 - 1}$
13. $z = \dfrac{\sin^{-1} \log_2 x + \sqrt{2x - y^2}}{\ln(x - y)}$
14. $z = \dfrac{\ln\left(x\sqrt{y - x}\right)}{xy - 1}$

15. $z = \dfrac{\sin^{-1}\left(x^2+y^2-2\right)-2}{x\sqrt{\sqrt{y}-x}}$

16. $z = \ln \dfrac{1-\sqrt{x^2-y^2}}{xy}$.

2 Level curves

Find the level curves for the following functions. Use also Matlab as a confirmation of the results obtained.

1. $z = x^2 + y^2$
2. $z = 2x + y$
3. $z = \dfrac{x}{y}$
4. $z = \ln \sqrt{\dfrac{y}{x}}$
5. $z = \dfrac{\sqrt{x}}{y}$
6. $z = e^{xy}$
7. $z = x + y$
8. $z = x^2 - y^2$
9. $z = \sqrt{xy}$
10. $z = \ln\left(x^2 + y\right)$
11. $z = \sin^{-1}(xy)$.

3 Gradient

Compute the gradient for the following functions.

1. $f(x, y) = x^2 - 3xy - 4y^2 - x + 2y + 1$
2. $f(x, y) = \dfrac{x-y}{x+1}$
3. $f(x, y) = \sqrt{x^2 - y^2}$
4. $f(x, y) = \dfrac{x}{\sqrt{x^2+y^2}}$

5. $f(x,y) = x\sqrt{y} + \dfrac{y}{\sqrt[3]{x}}$

6. $f(x,y) = \ln\left(x + \sqrt{x^2+y^2}\right)$

7. $f(x,y) = \ln \dfrac{\sqrt{x^2+y^2} - x}{\sqrt{x^2+y^2} + x}$

8. $f(x,y) = xy\ln(x+y)$

9. $f(x,y) = e^{-\frac{x}{y}}$

10. $f(x,y) = e^{xy(x^2+y^2)}$

Given the function $f(x,\, y,\, z) = \ln(xy + z)$, prove that $f'_x(1,\, 2,\, 0) = 1$, $f'_y(1,\, 2,\, 0) = \frac{1}{2}$, and $f'_z(1,\, 2,\, 0) = \frac{1}{2}$.

4 Unconstrained optimization

Determine the critical points of the following functions in their domain, specifying whether they are maximum, minimum or saddle points:

1. $f(x,y) = e^{x^2+y^2} - \dfrac{3}{2}x^2$

2. $f(x,y) = (x^2 - y^2)\log(xy)$

3. $f(x,y) = (x+y)e^{-(x^2+y^2)}$

4. $f(x,y) = \left(\dfrac{1}{y} + \dfrac{y}{x}\right)e^x$

5. $f(x,y) = y^3 + 3x^2y - 30y - 18x$

6. $f(x,y) = x^2\ln(1 + x^2 + y^2)$

7. $f(x,y) = \tan^{-1}(x^2 - y^2) - \dfrac{1}{3}x^3$

8. $f(x,y) = e^{(x-1)(y-2x)} + (y-2)^2$

9. $f(x,y) = 2\ln(x^2 + y^2 - 1) + x + 2y$

10. $f(x, y) = x \ln |x - y^2|$

11. $f(x, y) = \ln(1 + x^2 + 3y^2) - y^2$

12. $f(x, y) = (x + 2y)e^{-(x^2+y^2)}$

13. $f(x, y) = y \ln(2x + y)$

14. $f(x, y) = x^2 + e^{y^2-2x^2}$

15. $f(x, y) = x^2 + y^2 - x^2 \sin y^2$

16. $f(x, y) = \ln(1 + x^2) - x^2 + xy - y^2$

17. $f(x, y) = x^4 + (x - y)^3$

18. $f(x, y) = -x^3 y + 3x^2 y^2$.

5 Taylor polynomials

Write the second-degree Taylor polynomial for the following functions in the given point:

1. $f(x, y) = x^2 y^3, \quad \mathbf{x}_0 = (1,\ 1)$
2. $f(x, y) = \sin x \sin y, \quad \mathbf{x}_0 = (0,\ 0)$
3. $f(x, y) = x^2 + xy + 2y^2, \quad \mathbf{x}_0 = (1,\ -2)$
4. $f(x, y) = x^4 + y^4 - (x - y)^4, \quad \mathbf{x}_0 = (1,\ 0)$
5. $f(x, y) = e^{2x-3y}, \quad \mathbf{x}_0 = (1,\ 0)$
6. $f(x, y) = (x + y^2) \sin(x - y), \quad \mathbf{x}_0 = (0,\ 0)$
7. $f(x, y) = (x - 2y) \ln(xy), \quad \mathbf{x}_0 = (2,\ 1)$
8. $f(x, y) = x^5 y^3 - x^3 y^5, \quad \mathbf{x}_0 = (1,\ -1)$
9. $f(x, y) = xy \ln(xy^2) + x^2 y, \quad \mathbf{x}_0 = \left(1,\ e^{-\frac{3}{2}}\right)$.

For exercises 3–4 find also the critical point and discuss their nature.

6 Constrained optimization

Solve the following constrained optimization problems, where C denotes the constraint set:

1. $f(x, y) = x + y$, $C = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = 1\}$

   Sol: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ global maximum $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ global minimum

2. $f(x, y) = \sqrt{x^2 + y^2}$, $C = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = 9\}$

   Sol: $(0, \pm 3)$ global maxima $(\pm 3, 0)$ global minima

3. $f(x, y) = 2x^2 + y^2 - x$, $C = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \leq 1\}$

   Sol: $(-1, 0)$ global maximum $\left(\frac{1}{4}, 0\right)$ global minimum

4. $f(x, y) = 3x^2 + 4y^2 - 6x - 12$, $C = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 - 4 \leq 0\}$

   Sol: $(-2, 0)$ global maximum $(1, 0)$ global minimum

5. $f(x, y) = e^{xy}$, $C = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 - 1 \leq y \leq 3\}$

6. $f(x, y, z) = \ln\left(5x^2 + z^2 - y^2 + 6\right)$, $C = \{(x, y, z) \in \mathbb{R}^3 \,:\, 25x^2 + y^2 + z^2 \leq 4\}$

   Sol: $(0, 0, \pm 2)$ global maxima $(0 \pm 2, 0)$ global minimum

7. $f(x, y, z) = 3x+4y+11$, $C = \{(x, y, z) \in \mathbb{R}^3 \,:\, (x-2)^2 + (y-3)^2 + (z-4)^2 \leq 1\}$

   Sol: $(13/5, 19/5, 4/5)$ global maximum $(7/5, 11/5, 4)$ global minimum

8. $f(x, y, z) = z(x^2 - y^2)$, $C = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 - z^2 \leq 1, 0 \leq x^2 + y^2 \leq 2\}$

   Sol: $\left(\pm\sqrt{2}, 0, 1\right)$ global maxima $\left(0 \pm \sqrt{2}, 1\right)$ global minima

9. $f(x, y) = x^2 + y^2$, $C = \{(x, y) \in \mathbb{R}^2 \,:\, (x-1)^2 + (y-2)^2 - 20 = 0\}$

   Sol: $(3, 6)$ global maximum $(-1, -2)$ global minimum

10. Find the points on the circle $x^2 + y^2 = 80$ which are closest to and farthest from the point $(1, 2)$. Sol: $(-4, -8)$ and $(4, 8)$

11. $\min f(x, y) = 2y - x^2$, $C = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$.
    Sol: $(1, 0)$

12. Among all the triangles having perimeter $2p$, find the one whose area is maximum. Hint: use Heron's formula, stating that the area of a triangle with sides x, y, and z and perimeter $2p$ is: $\sqrt{p(p-x)(p-y)(p-z)}$.
    Sol: $x = y = z = 2p/3$