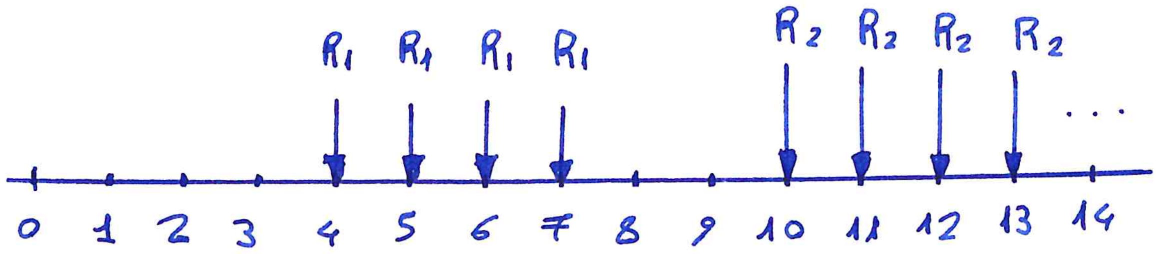


$$\sum_{k=0}^{m} x^k = \frac{1 - x^{m+1}}{1 - x}$$

$$W(\underline{R_1}, 0) = R_1 (1+i)^{-4} + R_1 (1+i)^{-5} +$$

$$+ R_1 (1+i)^{-6} + R_1 (1+i)^{-7} =$$

$$= R_1 (1+i)^{-4} \left[1 + (1+i)^{-1} + (1+i)^{-2} + \right.$$

$$\left. + (1+i)^{-3} \right] =$$

$$= R_1 (1+i)^{-4} \sum_{k=0}^{3} \left((1+i)^{-1} \right)^k =$$

$$= R_1 (1+i)^{-4} \frac{1 - (1+i)^{-4}}{1 - (1+i)^{-1}} =$$

$$= R_1 (1+i)^{-4} \frac{1 - (1+i)^{-4}}{1 - \frac{1}{1+i}} =$$

$$= R_1 (1+i)^{-3} \frac{1 - (1+i)^{-4}}{i}$$

$$W(\underline{R_2}, 0) = R_2 (1+i)^{-10} + R_2 (1+i)^{-11} +$$

$$+ R_2 (1+i)^{-12} + \dots =$$

$$= R_2 (1+i)^{-10} \left[1 + (1+i)^{-1} + (1+i)^{-2} + \dots \right] =$$

$$= R_2 (1+i)^{-10} \sum_{k=0}^{+\infty} \left((1+i)^{-1} \right)^k =$$

$$= R_2 (1+i)^{-10} \frac{1}{1 - (1+i)^{-1}} =$$

$$= R_2 (1+i)^{-10} \frac{1}{1 - \frac{1}{1+i}} =$$

$$= R_2 (1+i)^{-10} \frac{1}{\frac{\cancel{1} + i - \cancel{1}}{1+i}} =$$

$$= R_2 (1+i)^{-10} \frac{1+i}{i} = R_2 \frac{(1+i)^{-9}}{i}$$

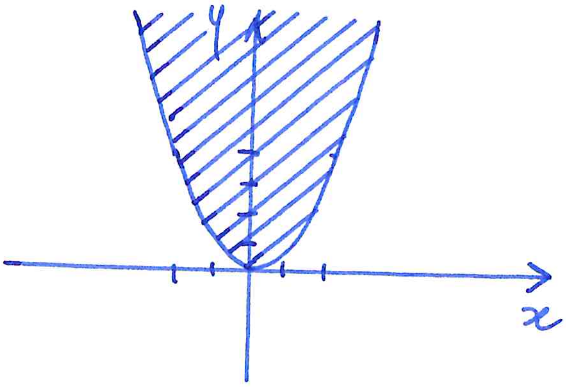
$$W(0) = W(\underline{R}_1, 0) + W(\underline{R}_2, 0) =$$

$$= R_1 (1+i)^{-3} \frac{1 - (1+i)^{-4}}{i} +$$

$$+ R_2 \frac{(1+i)^{-9}}{i}$$

$$W(7) = W(0)(1+i)^{7}$$

$$f(x,y) = \sqrt{y - x^2}$$



$$y - x^2 \geqslant 0 \qquad\qquad y \geqslant x^2$$

$$f'_x = \frac{1}{2\sqrt{y-x^2}}(-2x) = -\frac{1}{\sqrt{y-x^2}}$$

$$f'_y = \frac{1}{2\sqrt{y-x^2}}$$

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)(4-\lambda)+2\right] = 0$$

$$(\lambda-2)\left[(\lambda-1)(\lambda-4)+2\right] = 0$$

$$(\lambda-2)(\lambda^2-5\lambda+6) = 0$$

$$(\lambda-2)^2(\lambda-3) = 0$$

$$\lambda = 2 \quad \vee \quad \lambda = 3$$

$\underline{\lambda = 2}$

$(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{v} = \underline{0}$ $\qquad \underline{v} = (x, y, z)^T$

$(\underline{\underline{A}} - 2 \underline{\underline{I}}) \underline{v} = \underline{0}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} y = 0 \\ z = 0 \end{cases}$$

x is free to take any value.

Thus, $\underline{v} = (1, 0, 0)$ is a possible eigenvector associated to the eigenvalue $\lambda = 2$

$\underline{\lambda = 3}$

$(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{v} = \underline{0}$, $\qquad \underline{v} = (x, y, z)^T$

$(\underline{\underline{A}} - 3 \underline{\underline{I}}) \underline{v} = \underline{0}$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \sim$$

$$\sim \begin{cases} x - y = 0 \\ y + \frac{1}{2} z = 0 \end{cases} \qquad \begin{cases} x = y \\ y = -\frac{1}{2} z \end{cases} \qquad \begin{cases} x = k \\ y = k \\ z = -2k \end{cases} /$$

$k \in \mathbb{R}$

If, for example, $k = 1$, then

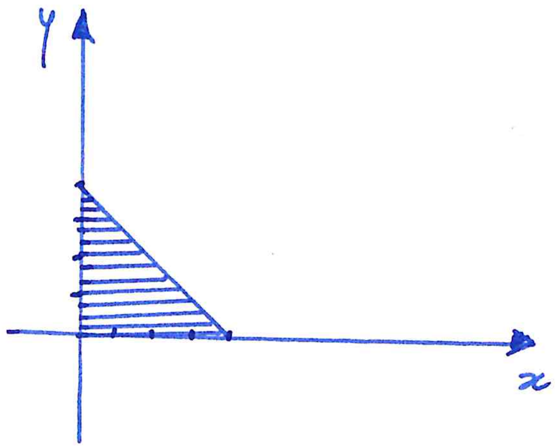
$\underline{v} = (1, 1, -2)^T$ is a possible

eigenvector associated to the eigenvalue $\lambda = 3$.

$$\frac{\underline{v}}{\|\underline{v}\|} = \frac{\underline{v}}{\sqrt{1^2 + 1^2 + (-2)^2}} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right)$$

$$f(x,y) = x^2 - xy + y$$

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \; x \geq 0, \; y \geq 0, \; x+y \leq 4\}$$



Ω is compact and f is continuous. Thus, by Weierstrass's theorem, global maximum and minimum exist.

METHOD 1: LAGRANGE MULTIPLIERS

$$\begin{cases} -x \leq 0 \\ -y \leq 0 \\ x+y \leq 4 \end{cases} \qquad \begin{cases} x+y \leq 4 \\ -x \leq 0 \\ -y \leq 0 \end{cases}$$

$$\underline{\underline{J}} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The rank of the matrix $\underline{\underline{J}}$ is always 2, as much as possible. Thus, qualification constraints are always satisfied.

$$\mathcal{L} = x^2 - xy + y - \lambda_1(x + y - 4) + \lambda_2 x + \lambda_3 y$$

$$\lambda_1(x + y - 4) = 0$$

$$\lambda_2 x = 0$$

$$\lambda_3 y = 0$$

$$x + y \leq 4, \quad x \geq 0, \quad y \geq 0$$

$\lambda_1, \lambda_2, \lambda_3 \geq 0$ for max. points

$\lambda_2, \lambda_2, \lambda_3 \leq 0$ for min. points

$$\nabla \mathcal{L} = \underline{0}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0: \quad 2x - y - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0: \quad -x + 1 - \lambda_1 + \lambda_3 = 0$$

<u>case 1: $\lambda_1 = \lambda_2 = \lambda_3 = 0$</u>

$$\begin{cases} 2x - y = 0 \\ -x + 1 = 0 \end{cases} \qquad \begin{cases} y = 2 \\ x = 1 \end{cases}$$

$x + y = 3 \leq 4$ ✓

$x = 1 \geq 0$ ✓

$y = 2 \geq 0$ ✓

$A(1, 2)$

case 2: $\lambda_1 = \lambda_2 = 0, \quad y = 0$

$$\begin{cases} 2x = 0 \\ -x + 1 + \lambda_3 = 0 \end{cases} \qquad \begin{cases} x = 0 \\ \lambda_3 = -1 \end{cases}$$

$x + y = 0 + 0 \leq 4 \quad \checkmark$

$x = 0 \geq 0 \quad \checkmark$

$y = 0 \geq 0 \quad \checkmark$

$\lambda_3 = -1 \leq 0$ $\qquad B(0, 0)$ candidate minimum

case 3: $\lambda_1 = 0, \quad x = 0, \quad \lambda_3 = 0$

$$\begin{cases} -y + \lambda_2 = 0 \\ 1 = 0 \end{cases}$$ impossible

case 4: $\lambda_1 = 0, \quad x = 0, \quad y = 0$

$$\begin{cases} -\cancel{\lambda_1} + \lambda_2 = 0 \\ 1 + \lambda_3 = 0 \end{cases} \qquad \begin{cases} \lambda_2 = 0 \\ \lambda_3 = -1 \end{cases}$$ same as case 2

case 5: $x + y - 4 = 0, \quad \lambda_2 = \lambda_3 = 0$

$$\begin{cases} x + y - 4 = 0 \\ 2x - y - \lambda_1 = 0 \\ -x + 1 - \lambda_1 = 0 \end{cases} \qquad \begin{cases} x = 4 - y \\ 2(4 - y) - y - \lambda_1 = 0 \\ -4 + y + 1 - \lambda_1 = 0 \end{cases}$$

$$\begin{cases} x = 4 - y \\ 8 - 2y - y - \lambda_1 = 0 \\ -3 + y - \lambda_1 = 0 \end{cases} \qquad \begin{cases} x = 4 - y \\ 8 - 3y - \lambda_1 = 0 \\ -3 + y - \lambda_1 = 0 \end{cases}$$

$$\begin{cases} x = 4 - y \\ \lambda_1 = 8 - 3y \\ -3 + y - 8 + 3y = 0 \end{cases} \qquad \begin{cases} x = 4 - y \\ \lambda_1 = 8 - 3y \\ 4y = 11 \end{cases} \qquad \begin{cases} x = \frac{5}{4} \\ \lambda_1 = -\frac{1}{4} < 0 \\ y = \frac{11}{4} \end{cases}$$

$x + y = \frac{5}{4} + \frac{11}{4} = \frac{16}{4} = 4 \leqslant 4 \quad \checkmark$

$x = \frac{5}{4} \geqslant 0 \quad \checkmark$

$y = \frac{11}{4} \geqslant 0 \quad \checkmark$

$\lambda_1 \leqslant 0$

$C\left(\frac{5}{4}, \frac{11}{4}\right)$ candidate minimum

<u>case 6: $x + y - 4 = 0, \quad \lambda_2 = 0, \quad y = 0$</u>

$$\begin{cases} x - 4 = 0 \\ 2x - \lambda_1 = 0 \\ -x + 1 - \lambda_1 + \lambda_3 = 0 \end{cases} \qquad \begin{cases} x = 4 \\ \lambda_1 = 8 \\ -4 + 1 - 8 + \lambda_3 = 0 \end{cases}$$

$$\begin{cases} x = 4 \\ \lambda_1 = 8 \geqslant 0 \\ \lambda_3 = 11 \geqslant 0 \end{cases}$$

$x + y = 4 + 0 = 4 \leqslant 0 \quad \checkmark$

$x = 4 \geqslant 0 \quad \checkmark$

$y = 0 \geqslant 0 \quad \checkmark$

$D(4, 0)$ candidate maximum

case 7: $x+y-4=0$, $x=0$, $\lambda_3=0$

$$\begin{cases} y-4=0 \\ -y-\lambda_1+\lambda_2=0 \\ 1-\lambda_1=0 \end{cases} \qquad \begin{cases} y=4 \\ -4-1+\lambda_2=0 \\ \lambda_1=1 \end{cases}$$

$$\begin{cases} y=4 \\ \lambda_2=5\geq 0 \\ \lambda_1=1\geq 0 \end{cases}$$

$x+y=0+4=4\leq 0$ ✓

$x=0\geq 0$

$y=4\geq 0$

$E(0,4)$ candidate maximum

case 8: $x+y-4=0$, $x=0$, $y=0$

$$\begin{cases} -4=0 \\ -\lambda_1+\lambda_2=0 \\ 1-\lambda_1+\lambda_2=0 \end{cases}$$ impossible

$f(x,y) = x^2-xy+y$

$f(A)=f(1,2)=1-2+2=1$

$\boxed{f(B)=f(0,0)=0}$ minimum

$f(C)=f\left(\frac{5}{4},\frac{11}{4}\right)=\frac{25}{16}-\frac{55}{16}+\frac{11}{4}=-\frac{30}{16}+\frac{11}{4}=$

$=-\frac{15}{8}+\frac{11}{4}=-\frac{15}{8}+\frac{22}{8}=\frac{7}{8}$

$\boxed{f(D)=f(4,0)=16}$ maximum

$f(E)=f(0,4)=4$

METHOD 2

Consider the unconstrained problem and look for minimum and maximum points also on the border of Ω

$$f(x,y) = x^2 - xy + y$$

$$\nabla f = (2x - y, \ -x + 1) = (0, 0)$$

$$\begin{cases} 2x - y = 0 \\ -x + 1 = 0 \end{cases} \qquad \begin{cases} y = 2 \\ x = 1 \end{cases} \qquad A(1, 2)$$

Let's check whether $A \in \Omega$

$x + y = 1 + 2 = 3 \leq 4$ ✓

$x = 1 \geq 0$ ✓

$y = 2 \geq 0$ ✓

Let's study the nature of the point A:

$f''_{xx} = 2 \qquad f''_{xy} = -1$

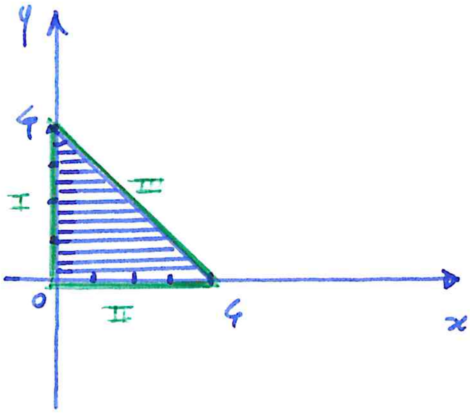
$f''_{yx} = -1 \qquad f''_{yy} = 0$

$$\underline{\underline{H}}f = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

$H_1 = 2 > 0$

$H_2 = \det \underline{\underline{H}}f = -1 < 0$

A: saddle point.



along side I

$x = 0, \quad 0 \leq y \leq 4$

$f(0, y) = 0^2 - 0 \cdot y + y = y$

Since $0 \leq y \leq 4$, the minimum is 0 and the maximum is 4.

Thus, $B(0,0)$ is a candidate minimum point and $E(0,4)$ is a candidate maximum point.

along side II

$y = 0, \quad 0 \leq x \leq 4$

$f(x, 0) = x^2$

Since $0 \leq x \leq 4$, the minimum is 0 and the maximum is reached when $x = 4$

Thus, $B(0, 0)$ is a candidate minimum point and $D(4, 0)$ is a candidate maximum point.

along side III

$x + y = 4$

$y = 4 - x$, $\qquad 0 \leqslant x \leqslant 4$

$q(x) := f(x, 4-x) = x^2 - x(4-x) + 4 - x =$

$= x^2 - 4x + x^2 + 4 - x = 2x^2 - 5x + 4$

$q'(x) = 0$; $4x - 5 = 0$; $x = \frac{5}{4}$

$q''(x) = 4 > 0$ candidate minimum

$y = 4 - x = 4 - \frac{5}{4} = \frac{11}{4}$

$C\left(\frac{5}{4}, \frac{11}{4}\right)$ candidate minimum

Also, $D(4,0)$ and $E(0,4)$ are candidate points.

The final conclusion presented in method 1 hold.