MATHEMATICAL FINANCE

martedì 21 febbraio 2023 09:26

$W(t_0)$: amount of money at time t_0

or $W(0)$ usually t_0 refers to today

1) you invest the money

$t_1 > t_0$

$W(t_1) > W(t_0)$

2) you borrow $W(t_0)$ today

$W(t_1)$: money to give back at time t_1

$W(t_1) > W(t_0)$

FIRST PROBLEM: to know $W(t_1)$ given $W(t_0)$

SECOND PROBLEM: $W(t_0)$ $W(t_1)$

I know I will receive $W(t_1)$ at t_1

What is the value of $W(t_1)$ today?

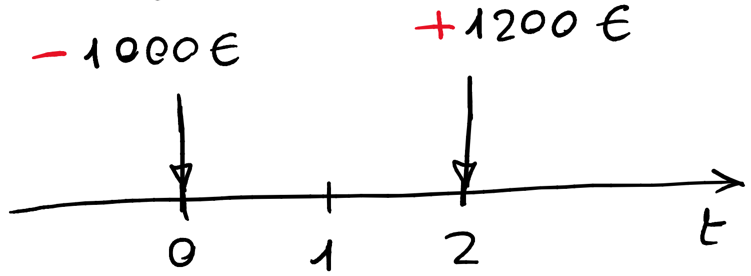
i.e. $W(t_0)$?

FINANCIAL OPERATION

EX1: INVEST today 1000 euro to receive in 2 years 1200 euro; **graphic representation**:

$W(0) = 1000$ €

$W(2) = 1200$ €



cash flow

1st   2nd

$$CF = \{ \underbrace{(-1000, +1200)}_{\text{vectors}} ; \underbrace{(0, 2)}_{\text{vectors}} \}$$

1 output   1 input

A vector $\vec{a} = \mathbf{a} = (a_1, a_2, a_3, a_4)$ tuple

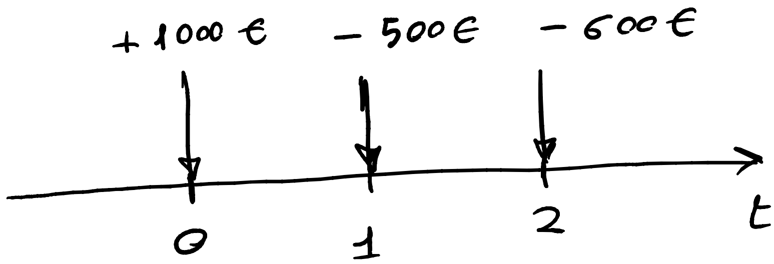
bold   first element

FINANCIAL OPERATIONS:

- simple: it is composed by one entry and one exit
- complex: it is a financial operation with more than 2 amounts

for example:

EX2: BORROW today 1000 euro, and pay 500 euro in one year and 600 euro in 2 years; flow-deadline representation:



$$OF = \{(+1000€, -500€, -600€), (0, 1, 2)\}$$

Investments

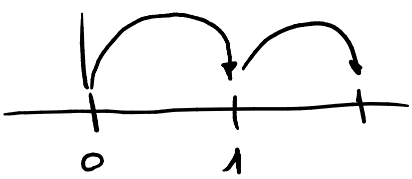
- simple interest
- compound interest

SIMPLE INTEREST

$W(0) \downarrow \qquad W(1) \downarrow$

i : interest rate refered to the scale of time in years.

$$W(1) = W(0) + i\,W(0) = W(0)(1 + 1 \cdot i)$$



now at year one, I invests the money again

$$W(2) = W(1) + i\,W(0) =$$

$$= W(0) + i\,W(0) + i\,W(0) =$$

$$= W(0) + 2i\,W(0) =$$

$$= W(0)(1 + 2i)$$

$$w(n) = w(0)(1 + n\,i) \qquad n \in \mathbb{N}$$

t

In general:

$$w(t) = w(0)(1 + it) \qquad t \in \mathbb{R}$$

If $w(0) \neq 0$

$$\frac{w(t)}{w(0)} = \frac{\cancel{w(0)}(1 + it)}{\cancel{w(0)}}$$

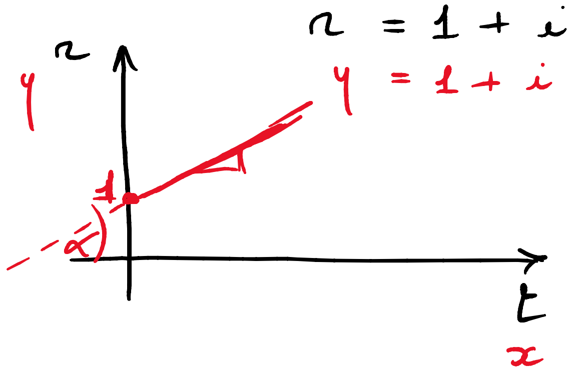
$$r(t) := \frac{w(t)}{w(0)} = 1 + it$$

growth factor

definition

$r = 1 + it$

$y = 1 + ix$



$$y = mx + q$$

$$m = tg\,\alpha$$

EX3: Consider the rule of simple interest. If today we invest 2000 euro at an interest rate (annual) of 10%=0.1, then in 3 years we will receive

$$W(3) = W(0)(1 + i\,3) =$$

$$= 2000€\,(1 + 0.1 \cdot 3) = 2600€$$

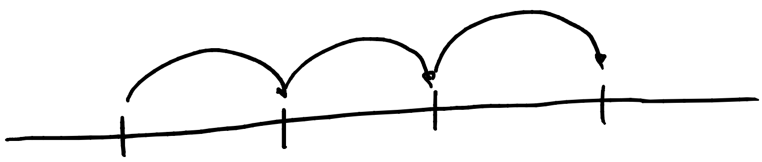
A person borrows \$2500 at 8% interest per year. Find the accumulated amount of money in 90 days.

$$\frac{90}{365} \qquad \underbrace{\frac{90}{360}}$$

$$w(90 \text{ days}) = W(0)\Big(1 + \underbrace{i}_{\text{years}} \cdot 90 \text{ days}\Big) =$$

$$= 2500\,\$\,\Big(1 + 0{,}08 \cdot \underbrace{\frac{90}{360}}_{\text{years}}\Big) =$$

$$= 2550\,\$$$



i_m : the interest rate referring to the $\underline{m\text{-th}}$ part of the year

For instance, if I invest every month, I will use i_{12}

$$w(t) = w(0)\,(1 + i_{12}\, t)$$

time in months

If I invest every 2 months, I will use i_6

equivalent interest rates

1 year

6 bimesters

$$W(t) = W(0)\,(1 + i\, t)$$

years

in 1 year, there are m m-th parts of the year

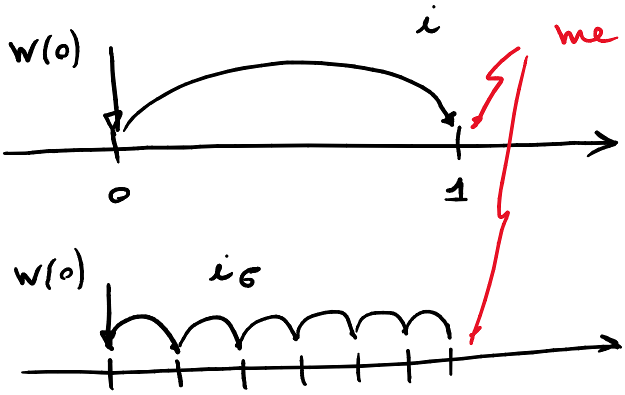
in a year

$$W(t) = W(0)\,(1 + i_m\, t \cdot m)$$

$$W(1) = W(0)\,(1 + i\, 1)$$

I change unit of time and I think in bimesters

i_m is equivalent to i if, when applied to the same initial value for the same period of time, they produce the same final value.



we get the money

$$W(1) = W(0)\,(1 + i\, 1)$$

year

$$W(1) = W(0)\,(1 + i_6\, 6)$$

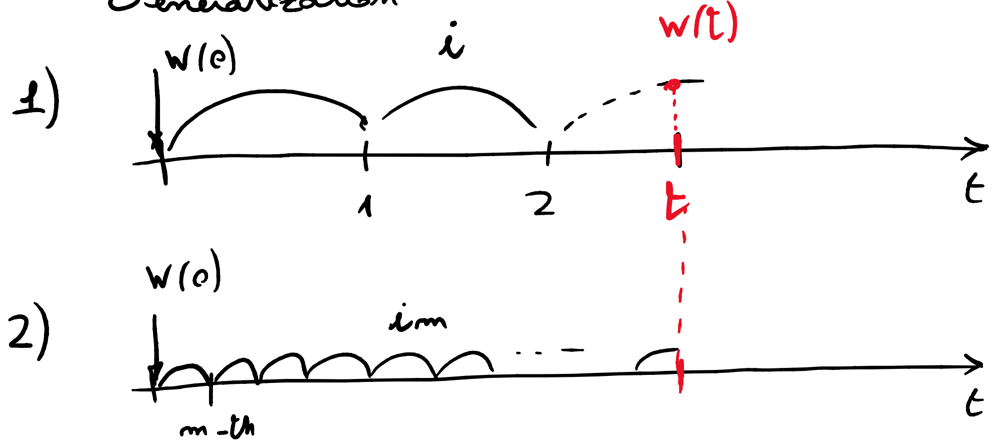
$$\cancel{W(0)}\,(1 + i\, 1) = \cancel{W(0)}\,(1 + i_6 \cdot 6)$$

$$\cancel{1} + i_1 = \cancel{1} + i_6 \cdot 6$$

$$i \cdot 1 = i_6 \cdot 6$$

$$i_6 = \frac{i}{6}$$

Generalization



1) $$w(t) = w(0)\,(1 + i t)$$

2) $$w(t) = w(0)\,(1 + i_m\, t \cdot m)$$

$$\cancel{w(0)}\,(1 + i t) = \cancel{w(0)}\,(1 + i_m \cdot t \cdot m)$$

$$\cancel{1} + i t = \cancel{1} + i_m \cdot t \cdot m$$

$$i \cancel{t} = i_m \cdot \cancel{t} \cdot m \qquad t \neq 0$$

$$\boxed{i_m = \frac{i}{m}} \qquad (1)$$

$$i = i_m \cdot m \qquad (2)$$

If I set $m = n$ in

$$i = i_m \cdot m \qquad (3)$$

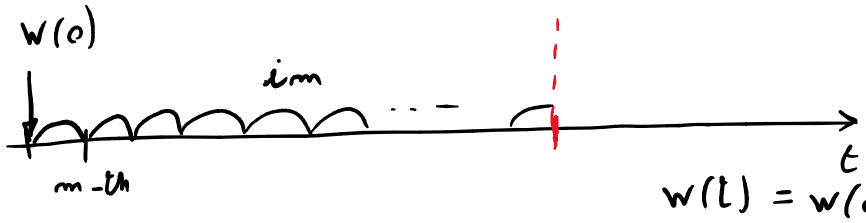
1 plug (3) into (1)

$$i_m = \frac{i_m \cdot m}{m}$$

$$\boxed{i_m \cdot m = i_m \cdot m} \qquad (4)$$

or

2)



$$w(t) = w(0)\left(1 + i_m \cdot m\, t\right)$$

$$w(t) = w(0)\left(1 + i_m \cdot m \cdot t\right)$$

$$i_2 = 5\%$$

$$i_6 = ?$$

$$i_6 \cdot 6 = i_2 \cdot 2$$

$$i_6 = \frac{i_2 \cdot 2}{6} = \frac{0{,}05 \cdot 2}{6} = 1{,}67\%$$