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linear algebra:

- systems of equations (Gaussian elimination)
    - Vector, matrixs (inverse matrix, transpose, symmetric matrix)

Adams, Essex

CALCULUS
- limits
- **Derivatives**
- (Taylor series)

$$\infty$$

$$W(t) = W(0)\,(1 + i t) \leftarrow$$

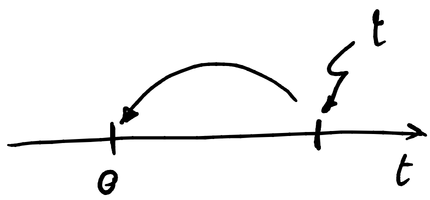
$$i_m \cdot m = i_m \cdot m$$

$W(0)$: present value or principal

$W(0) \neq 0$

future value

$$W(0) = \underbrace{\frac{1}{1 + i t}}_{<1} W(t)$$

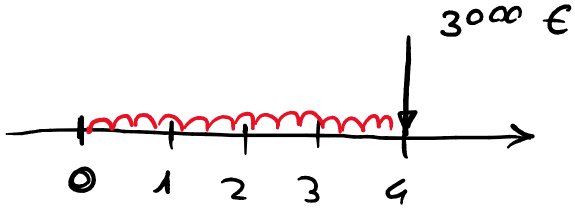


$$W(t) > W(0)$$

discount factor

$V(t)$

EX6: Bob holds a promissory note of 3000 euro at 4 years, but he wants to discount it to obtain today its present value for liquidity purpose. The bank applies the simple interest rule with a monthly interest rate of 1%.



% $\quad \frac{1}{100}$

$$i_{12} \cdot 12 = i \cdot 1$$

$$i = i_{12} \cdot 12 = 12 \cdot \frac{1}{100} = 0{,}12$$

$$W(4) = W(0)\,(1 + i \cdot 4)$$

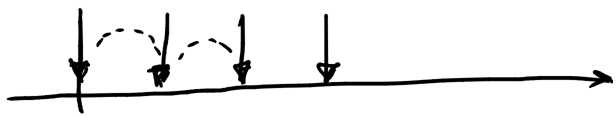
$$W(0) = \frac{W(4)}{1 + i4} = \frac{3000\text{ €}}{1 + 0{,}12 \cdot 4} = 2027\text{ €}$$

1e3 $\quad 10^3$

2e-1 $\quad 2 \cdot 10^{-1} = 0{,}2$

∞

COMPOUND INTEREST



SIMPLE

$$W(1) = W(0) + i\,W(0)$$
$$W(2) = W(1) + i\,W(0)$$
$$W(3) = W(2) + i\,W(0)$$

COMPOUND

$$W(1) = W(0) + i\,W(0) = W(0)(1+i)$$
$$W(2) = W(1) + i\,W(1) =$$
$$= W(1)\,(1 + i) =$$
$$= W(0)\,(1+i)^2$$

$$W(3) = W(2) + i\,W(2) =$$

$$= W(2)\,(1+i) =$$

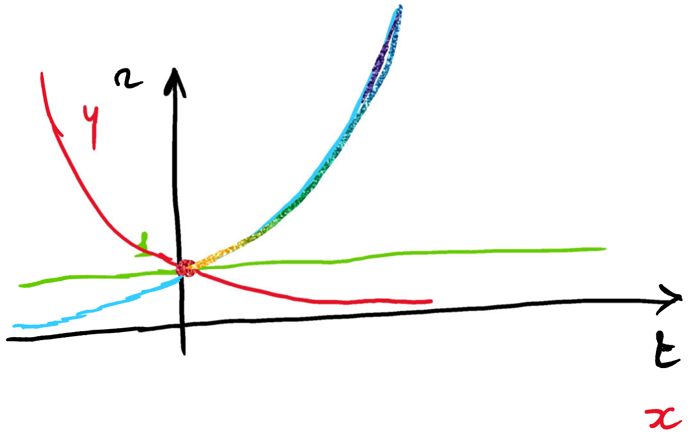
$$= W(0)\,(1+i)^3$$

In general

$$W(n) = W(0)\,(1+i)^n \qquad n \in \mathbb{N}$$

$$W(t) = W(0)\,(1+i)^t \qquad t \in \mathbb{R}$$

growth factor $r(t) := \dfrac{W(t)}{W(0)} = (1+i)^t$



$$r = (1+i)^t$$

$$y = \underbrace{(1+i)}^{x}$$

$$a > 1$$

$$y = a^x$$

base

$$y = a^x \qquad \begin{cases} \mathrm{I} & a > 1 \\ \mathrm{II} & a = 1 \\ \mathrm{III} & 0 < a < 1 \end{cases} \qquad a > 0$$

II

$$y = 1^x = 1$$

I $\quad a > 1$

for instance $\quad a = 2 \qquad y = 2^x$

$$x = 0 \qquad y = 2^0 = 1$$

$$x = 1 \qquad y = 2^1 = 2$$

III $\quad 0 < a < 1$

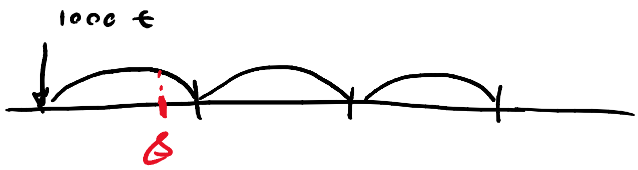
$$a = \frac{1}{2} \qquad y = \left(\frac{1}{2}\right)^x$$

$$x = 0 \qquad y = \left(\frac{1}{2}\right)^0 = 1$$

$$x = -1 \qquad y = \left(\frac{1}{2}\right)^{-1} = 2$$

EX7: Let i=10%. Then with the exponential rule 1000 euro today is equivalent, after 8 months, to

$$w(0) = 1000\ €$$



$$1 \text{ year} = 12 \text{ months}$$

$$1 \text{ month} = \frac{1}{12} \text{ year}$$

$$8 \text{ months} = \frac{8}{12} \text{ years.}$$

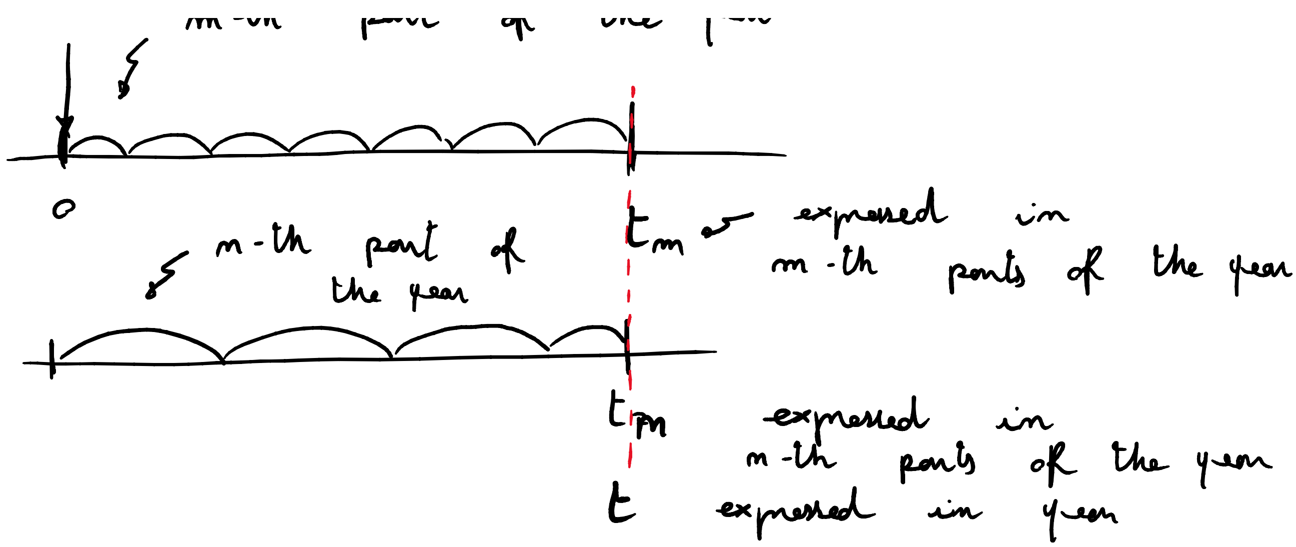
$$w(t) = w(0)\,(1+i)^t$$

$$w(8 \text{ months}) = 1000\ €\ (1+0,1)^{\frac{8}{12}} \cong 1065,60\ €$$

NO $\quad 1000 \dfrac{(1+0,1)^8}{12}$

$$i_m \cdot m = i_n \cdot n \qquad \text{simple}$$

$w(0)$ $\quad$ m-th part of the year



$t = 1$ year

semesters $t_2 = 2$
months $t_{12} = 12$

i_m is equivalent to i_n if investing the same capital $W(0)$ I get the same amount of money after a time t.

$$W(t_m) = W(t_n)$$

$$\cancel{W(0)}\,(1 + i_m)^{t_m} = \cancel{W(0)}\,(1 + i_n)^{t_n}$$

$$1 \text{ year} = \frac{1}{4} \cdot \underset{t_{12}}{12 \text{ month}}$$

$$t = t_m \cdot m$$

$$(1 + i_m)^{t\,m} = (1 + i_M)^{t\,M}$$

$$(1 + i_m)^{\cancel{t}\cdot m} = (1 + i_M)^{\cancel{t}\cdot M}$$

$$(1 + i_m)^{\frac{m}{m}} = (1 + i_M)^{\frac{M}{m}}$$

$$1 + i_m = (1 + i_M)^{\frac{M}{m}}$$

$$\boxed{i_m = (1 + i_M)^{\frac{M}{m}} - 1}$$

EX9: Let w(0)=2000, i2=0.05, t=20 months and consider the exponential rule. To calculate the future values the following procedures are equivalent.

i_2 every semester

I

$$W(20 \text{ months}) = W(0)\,(1 + i_2)^{\frac{20}{6}} =$$

$$= 2000\,€\,(1 + 0{,}05)^{\frac{20}{6}} \cong 2353{,}20\,€$$

II $\quad i_2 \longrightarrow i_{12}$

$$\boxed{i_m = (1 + i_M)^{\frac{M}{m}} - 1}$$

$$i_{12} = (1 + i_2)^{\frac{2}{12}} - 1 = 0{,}0082$$

$\boxed{0{,}05}$

$i_2 \quad \boxed{0{,}05}$

$$w(20 \text{ months}) = 2000\text{ €}\,(1 + 0{,}0082)^{20} \cong 2353{,}20\text{ €}$$

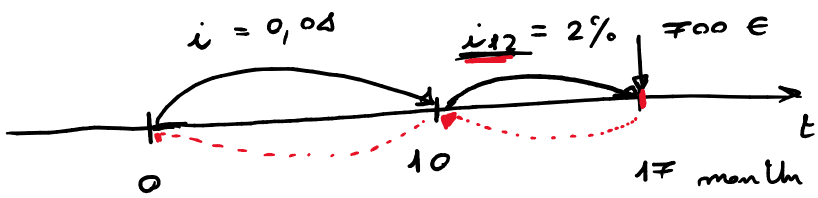
$$w(t) = w(0)\,\underbrace{(1+i)^t}_{\text{growth factor} \quad a(t)}$$

$$w(0) = \underbrace{\frac{1}{(1+i)^t}}\, w(t) = \underbrace{(1+i)^{-t}}\, w(t)$$

discount factor

$$v(t) := \frac{1}{a(t)} = (1+i)^{-t}$$

EX11: The amount of 700 euro will be disposable in 17 months. We wants to determine its present value with the exponential rule by considering that: (1) during the first 10 months it is given i=0.08.(2) during the last period it is applied the monthly interest rate 2%



$w(0) = ?$

$$\underline{i'_{12}} = (1+i)^{\frac{1}{12}} - 1 = (1{,}08)^{\frac{1}{12}} - 1 = 0.0064$$

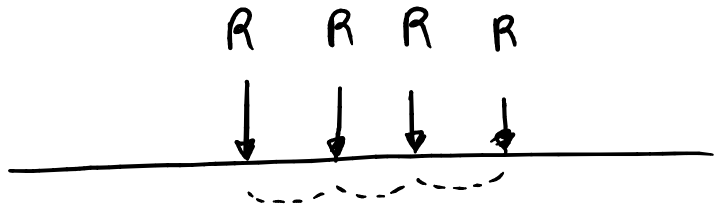
$$w(10) = w(17)\,(1 + i_{12})^{-7} \leftarrow$$

$$w(0) = w(10)\,(1 + i'_{12})^{-10} =$$

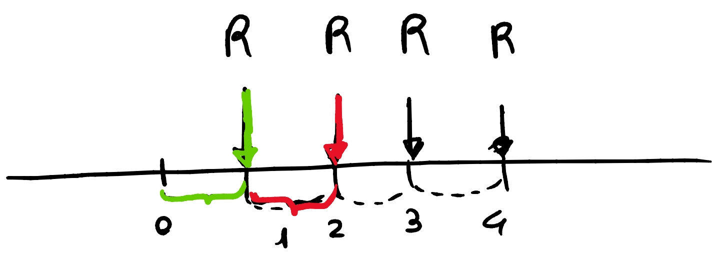
$= W(17)(1+i_{12})^{-7}(1+i_{\frac{1}{2}})^{-10} \cong 571,54$ €

ANNUITY

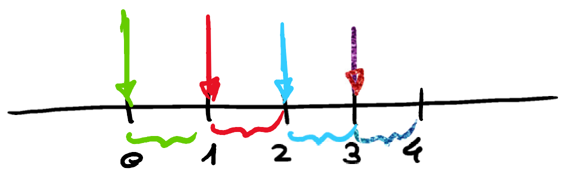
A sequence of regular finite payments



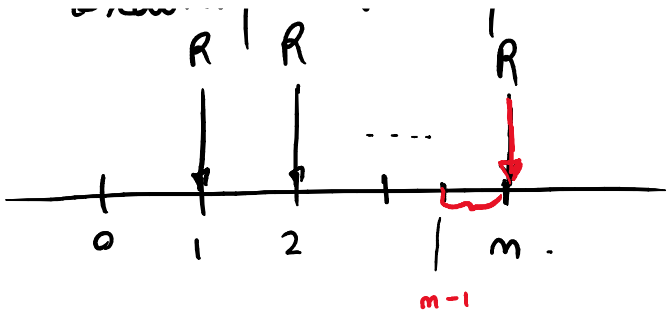
ordinary annuity or annuity immediate
if the payments are made at the end
of the payment periods



annuity due: if the payments are made
at the beginning of the payment periods

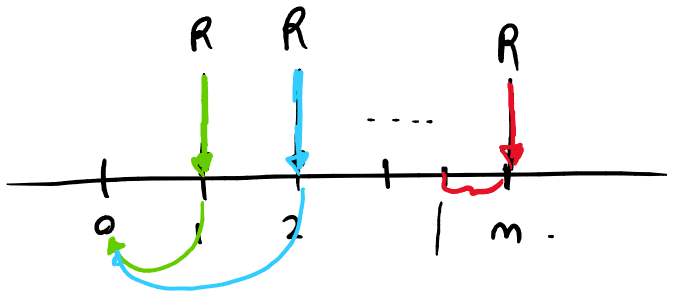


ordinary annuity



compound interest rule

What is the present value $w(R, 0)$ of the annuity?



$$w(1) = w(0)\,(1+i)$$

$$R = w(0)\,(1+i)$$

$$R\,(1+i)^{-1}$$

$$w(k) = w(0)\,(1+i)^{k}$$

$$R = w(0)\,(1+i)^{k}$$

$$w(0) = R\,(1+i)^{-k}$$

$$w(R, 0) = R\,(1+i)^{-1} + R\,(1+i)^{-2} + \dots +$$

$$+ R\,(1+i)^{-m}$$

$$x := (1+i)^{-1}$$

$$x := (1+i)^{-1}$$

$$w(R,0) = R\,(\underbrace{x + x^2 + \dots + x^n})$$

geometric sum

$$S_n := 1 + x + x^2 + \dots + x^n$$

$$S_n = 1 + x\,(1 + x + \dots + x^{n-1})$$

$$S_n = 1 + x\,(\underbrace{1 + x + \dots + x^{n-1} + x^n}_{S_n} - x^n)$$

$$S_n = 1 + x\,(S_n - x^n)$$

$$S_n = 1 + x S_n - x^{n+1}$$

$$S_n - x S_n = 1 - x^{n+1}$$

$$(1-x)\,S_n = 1 - x^{n+1} \qquad x \neq 1$$

$$S_n = \frac{1 - x^{n+1}}{1-x}$$

$$w(R,0) = R\,(\underbrace{x + x^2 + \dots + x^n}) =$$

$$= R\,x\,(\underbrace{1 + x + \dots + x^{n-1}}_{S_{n-1}}) =$$

$$= R\,x\,\frac{1 - x^n}{1 - x} =$$

$$= R\,\frac{1 - x^n}{\frac{1-x}{x}} =$$

$$= R\,\frac{1 - x^n}{\frac{1}{x} - 1} =$$

$$x = (1+i)^{-1} =$$

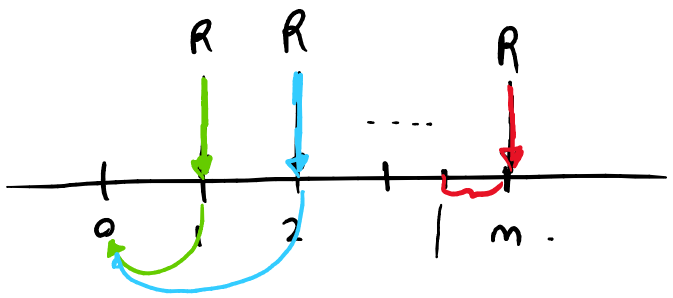
$$= \frac{1}{1+i}$$

$$\frac{1}{x} = 1 + i$$

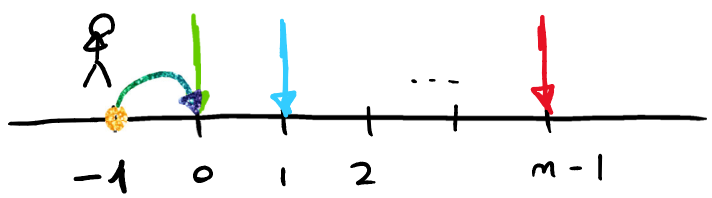
$$= \underbrace{R\,\frac{1 - (1+i)^{-n}}{i}}_{a_{\overline{n}|i}}$$

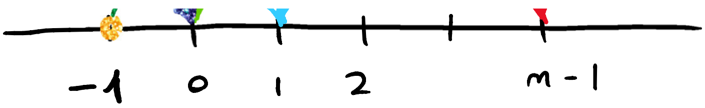
a angle n at i

ordinary annuity



annuity due





From time -1, we see an ordinary annuity

$$W(R,-1) = R\, a_{\overline{m}|i}$$

$$W(R,0) = W(R,-1)(1+i) =$$

$$= R\, a_{\overline{m}|i}(1+i) =$$

$$= R \underbrace{\frac{1-(1+i)^{-m}}{i}(1+i)}_{\ddot{a}_{m|i}}$$

$\ddot{a}_{m|i}$ a dots angle m at i