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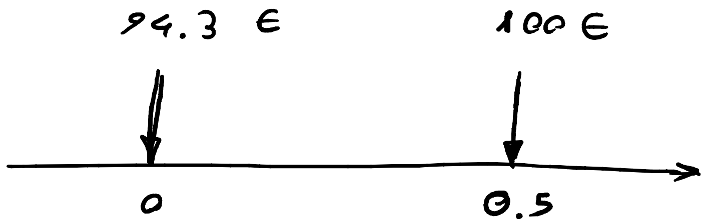
EX26: consider a zero coupon bond (ZCB) that is quoted 94.3, ending in 6 months, nominal value (rembursmente value) 100.



interest rate = ?

$$W(t) = W(0)\,(1+i)^{t}$$

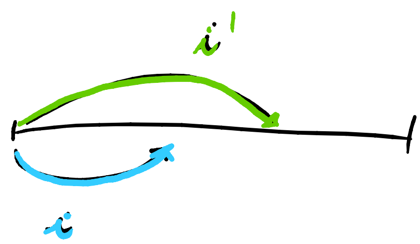
$$100\,\cancel{€} = 94.3\,\cancel{€}\,(1+i)^{\frac{1}{2}}$$

$$\frac{100}{94.3} = \sqrt{1+i}$$

$$\left(\frac{100}{94.3}\right)^2 = 1 + i$$

$$i = \left(\frac{100}{94.3}\right)^2 - 1 = 0.1245$$

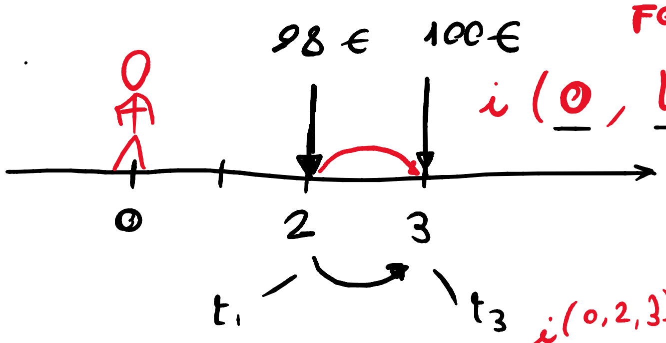
SPOT INTEREST RATES



The interest rate can vary in time

$i(0, t)$: the interest rate from time 0 to time t

EX28: today we sign a contract to buy a ZCB at time t=2 (years) at price 98 whose deadline is time t=3 (nominal rembursment value is 100).



FORWARD INTEREST RATE

$i(\underline{0}, \underline{t-1}, \underline{t})$

$i(\underline{0}, 2, 3)$

interest rate computed today (time 0) relative to an investment from time 2 to time 3

$i(0,2,3)$

$$W(t_1) = W(0)(1+i)^{t_1}$$

$$W(t_2) = W(0)(1+i)^{t_2}$$

$$\frac{W(t_2)}{W(t_1)} = \frac{\cancel{W(0)}(1+i)^{t_2}}{\cancel{W(0)}(1+i)^{t_1}}$$

$$\frac{W(t_2)}{W(t_1)} = (1+i)^{t_2 - t_1}$$

$$W(t_2) = W(t_1)(1+i)^{t_2 - t_1}$$

$$W(3) = W(2)(1+i)^{3-2}$$

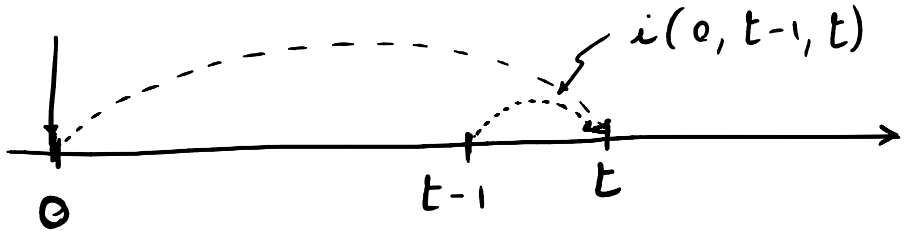
$$100 \cancel{€} = 98 \cancel{€} (1+i)^1$$

$$\frac{100}{98} = 1 + i$$

$$i = \frac{100}{98} - 1 = 0.0204$$

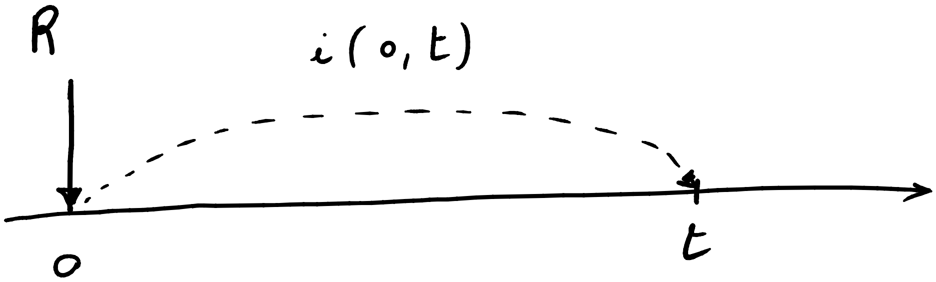
∞

$i(0,t)$: spot interest rate



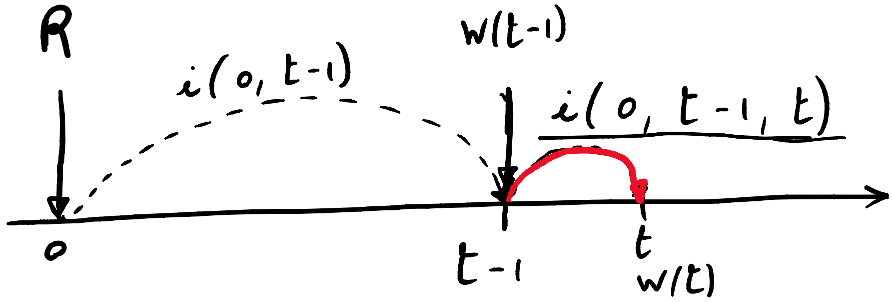
forward interest rate

Ⓐ



$$w(t) = R\,(1 + i(0,t))^{t}$$

Ⓑ



$$w(t) = \overbrace{w(t-1)}\,(1 + \underline{i(0,\ t-1,\ t)})^{1} \quad \longleftarrow$$

$$w(t) = R\,(1 + i(0, t-1))^{t-1}\,(1 + i(0,\ t-1,\ t))$$

$$\cancel{R}\,(1 + i(0,t))^{t} = \cancel{R}\,(1 + i(0, t-1))^{t-1}\,(1 + i(0,\ t-1,\ t))$$

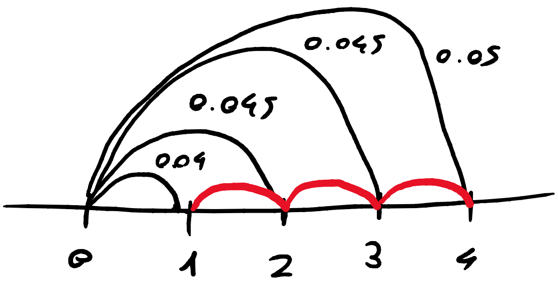
$$\cancel{R}\,(1+i(0,t))^t = \cancel{R}\,(1+i(0,t-1))\,(1+i(0,t-1,t))$$

$$\frac{(1+i(0,t))^t}{(1+i(0,t-1))^{t-1}} = \frac{\cancel{(1+i(0,t-1))^{t-1}}\,(1+\underline{i(0,t-1,t)})}{\cancel{(1+i(0,t-1))^{t-1}}}$$

$$i(0,t-1,t) = \frac{(1+i(0,t))^t}{(1+i(0,t-1))^{t-1}} - 1$$

EX29: Suppose the spot rates of interest for investment horizons of 1, 2, 3 and 4 years are, respectively, 4%, 4.5%, 4.5%, and 5%. Calculate the forward rates of interest for t = 1, 2, 3 and 4.

$$i(0,1) = 0.04$$
$$i(0,2) = 0.045$$
$$i(0,3) = 0.045$$
$$i(0,4) = 0.05$$



$$i(0,t-1,t) = \frac{(1+i(0,t))^t}{(1+i(0,t-1))^{t-1}} - 1 \leftarrow$$

If $t = 2$

$$i(0,1,2) = \frac{(1+i(0,2))^2}{(1+i(0,1))^1} - 1 = \frac{(1+0.045)^2}{1+0.04} - 1 =$$

$$= \frac{1.045^2}{1.04} - 1 = 5.0024\ \%$$

$t = 3$

$$i(0,2,3) = \frac{(1+i(0,3))^3}{(1+i(0,2))^2} - 1 = \frac{1.045^{\cancel{3}}}{\cancel{1.045^2}} - 1 =$$

$$= 0.045 = 4.5\%$$

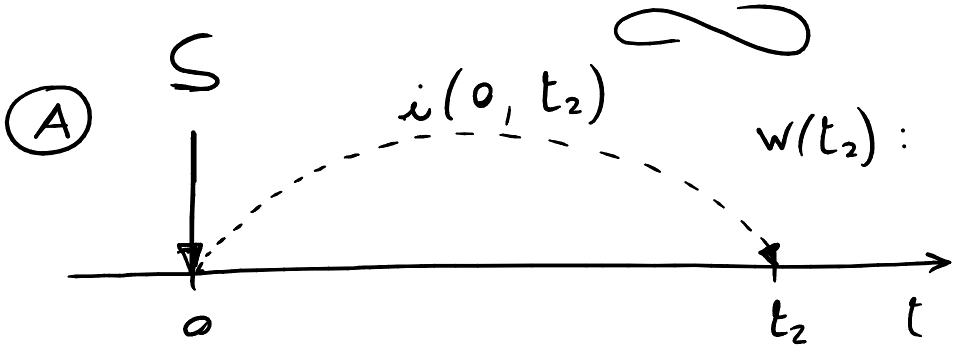
$$i(0,t-1,t) = \frac{(1+i(0,t))^t}{(1+i(0,t-1))^{t-1}} - 1$$

$t = 4$

$$i(0,3,4) = \frac{(1+i(0,4))^4}{(1+i(0,3))^3} - 1 = \frac{1.05^4}{1.045^3} - 1 \cong$$

$$\cong 6.5\%$$

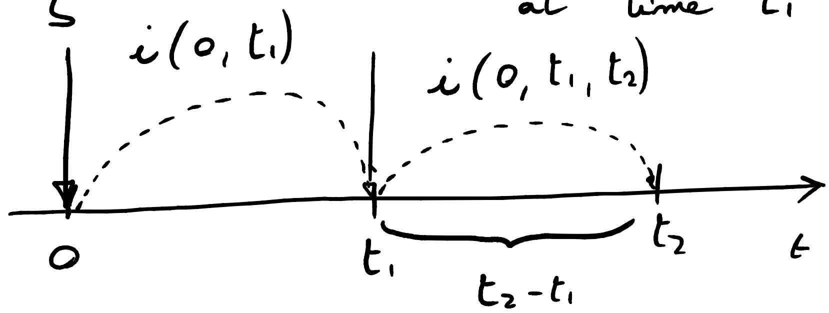
Ⓐ



$w(t_2)$: collected money at time t_2

Ⓑ

$w(t_1)$: money collected and reinvested at time t_1



Ⓐ $w(t_2) = S\,(1 + i(0,t_2))^{t_2}$ . . .

Ⓐ $w(t_2) = S\,(1 + i(0, t_2))^{t_2}$

Ⓑ $w(t_2) = w(t_1)\,(1 + i(0, t_1, t_2))^{t_2 - t_1} =$

$$= S\,(1 + i(0, t_1))^{t_1}\,(1 + i(0, t_1, t_2))^{t_2 - t_1}$$

$$\cancel{S}(1 + i(0, t_2))^{t_2} = \cancel{S}\,\underbrace{(1 + i(0, t_1))^{t_1}}\,(1 + \underline{i(0, t_1, t_2)})^{t_2 - t_1}$$

$$\frac{(1 + i(0, t_2))^{t_2}}{(1 + i(0, t_1))^{t_1}} = (1 + i(0, t_1, t_2))^{t_2 - t_1}$$

$$(\ .....\ ) = x^3$$

$$(....)^{\frac{1}{3}} = \sqrt[3]{...} = x$$

$$\left[\frac{(1 + i(0, t_2))^{t_2}}{(1 + i(0, t_1))^{t_1}}\right]^{\frac{1}{t_2 - t_1}} = (1 + i(0, t_1, t_2))^{\frac{t_2 - t_1}{t_2 - t_1}}$$

$$i(0, t_1, t_2) = \left[\frac{(1 + i(0, t_2))^{t_2}}{(1 + i(0, t_1))^{t_1}}\right]^{\frac{1}{t_2 - t_1}}$$

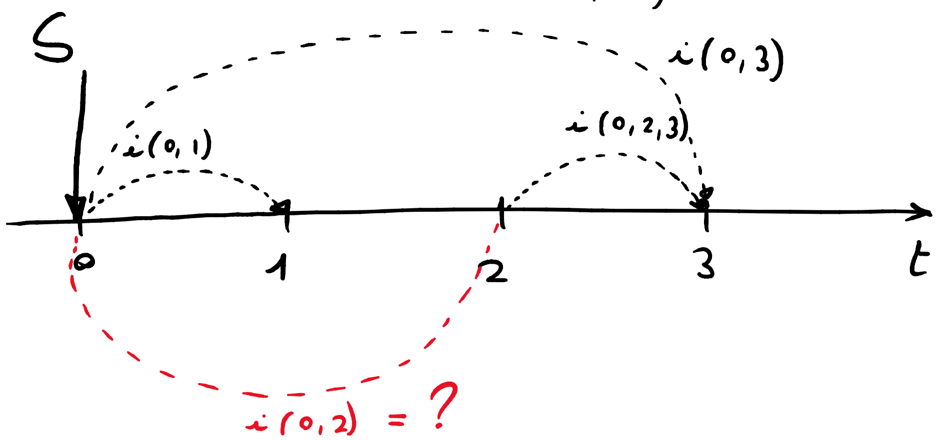
multi - period forward rate

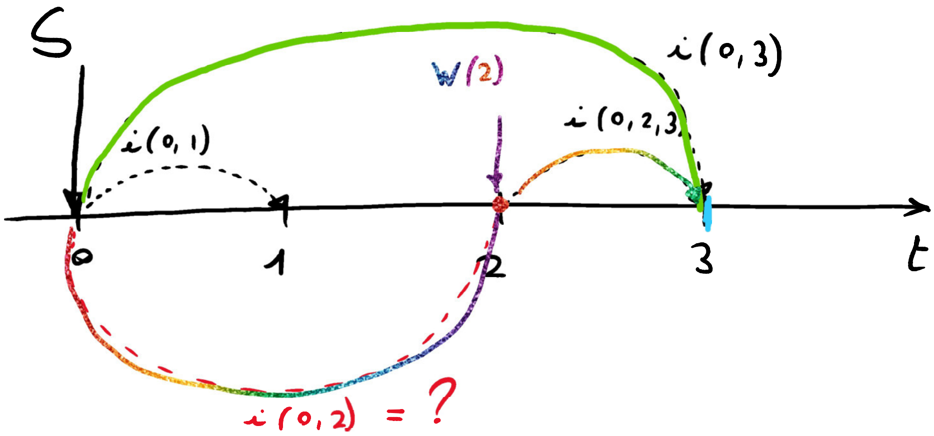
EX30: Let be given i(0,1)=0.075, i(0,2,3)=0.035, i(0,3)=0.05.

The term structure of the spot rates of interest can be obtained.

Compute : $i(0,1,3)$, $i(0,1,2)$,

and $i(0,2)$





$$W(3) = S\,(1 + i(0,3))^{3}$$

$$W(3) = W(2)\,(1 + i(0,2,3))^{3-2} =$$

$$= S\,(1 + i(0,2))^{2}\,(1 + i(0,2,3))$$

$$= S\,(1 + \underline{i(0,2)})^2\,(1 + i(0,2,3))$$

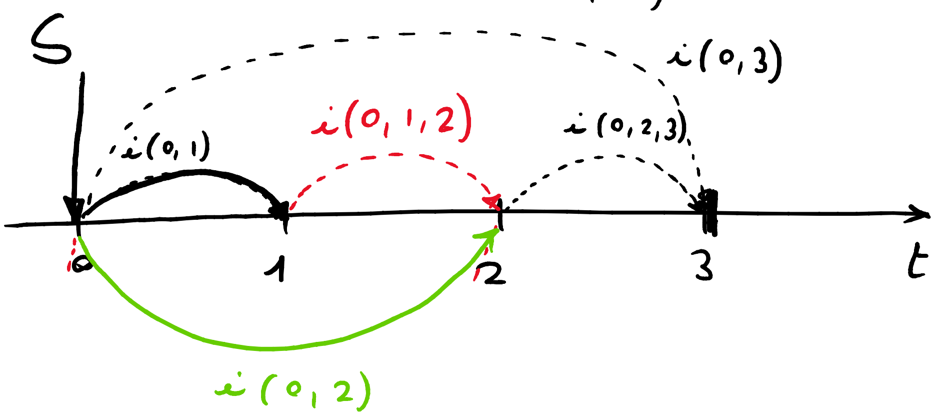
$$\cancel{S}\,(1 + i(0,3))^3 = \cancel{S}\,(1 + i(0,2))^2\,(1 + i(0,2,3))$$

$$1.05^3 = (1 + i(0,2))^2\; 1.035$$

$$(1 + i(0,2))^2 = \frac{1.05^3}{1.035}$$

$$i(0,2) = \sqrt{\frac{1.05^3}{1.035}} - 1 \cong 0.057$$

and $i(0,2)$



$$W(2) = S\,(1 + i(0,2))^2$$

$$W(2) = S\,(1 + i(0,1))^1\,(1 + \underline{i(0,1,2)})^1$$

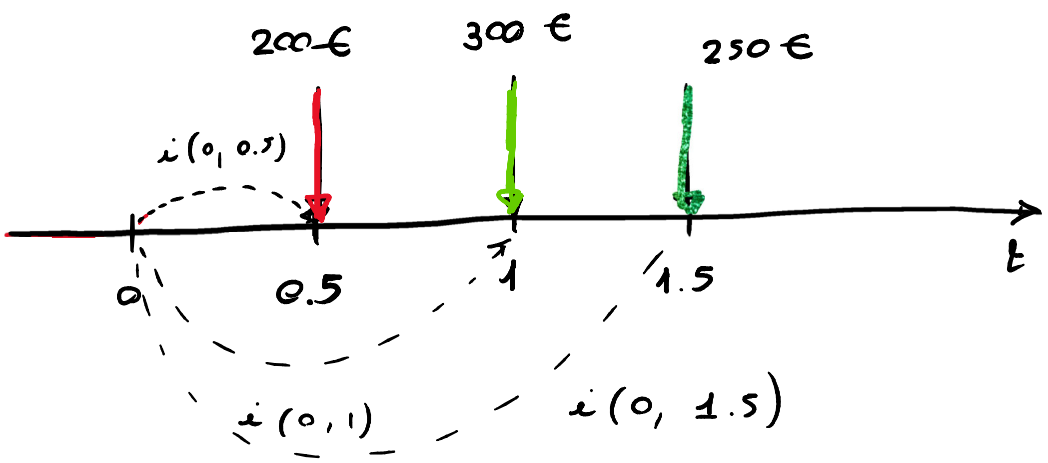
or

$$W(3) = S\,(1 + i(\ldots))^3$$

$$W(3) = S\left(1 + i(0,3)\right)^{3}$$

$$W(3) = S\left(1 + i(0,1)\right)^{1}\left(1 + \underline{i(0,1,2)}\right)^{1}\left(1 + i(0,2,3)\right)^{1}$$

EX31: let i(0,0.5)=0.04, i(0,1)=0.045 and i(0,1.5)=0.05, and recall that these are annual interest rates. Consider the following payments: 200 euro in 6 months, 300 euro in one year, 250 euro in 3 semesters. Then the present value of the cash flow is given by:



$$W(t) = W(0)(1+i)^{t}$$

$$W(0) = W(t)(1+i)^{-t}$$

$$W(0) = S = 200\,€\left(1 + i(0, 0.5)\right)^{-0.5} +$$

$$+ 300\,€\left(1 + i(0,1)\right)^{-1} +$$

$$+ 250\,€\left(1 + i(0, 1.5)\right)^{-1.5} =$$

$$= 200\,€ \; 1.04^{-0.5} + 300\,€ \; 1.045^{-1} +$$

$$+ 250\,€ \; 1.05^{-1.5} \cong 715.56\,€$$