giovedì 2 marzo 2023 08:37

$\underline{V} = \vec{V} = [1, 2, 3, 4]$

multiplication by a $\underline{\text{scalar}}$ $a \in \mathbb{R}$ (or $\mathbb{C}$)

$a\vec{V} = [aV_1, aV_2, \dots, aV_n]$

$2\underline{V} = [2, 4, 6, 8]$

$V_1 = [1, 2, 3, 4]$

$\underline{u}_1 = [1, 1, 1, 1]$

$\underline{u}_1 + \underline{V}_1 = [1+1, 2+1, 3+1, 4+1] =$
$= [2, 3, 4, 5]$

WEIGHTED AVERAGE

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_N}{N}$$

$$\min\{x_1, \dots, x_N\} \leq \bar{x} \leq \max\{x_1, \dots, x_N\}$$

weighted average

$$\bar{x}_V = \frac{V_1 x_1 + V_2 x_2 + \dots + V_N x_N}{V_1 + V_2 + \dots + V_N}$$

The simple average is a particular case

$V_1 = V_2 = \dots = V_N = 1$

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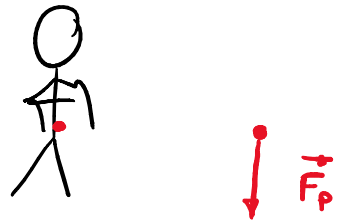
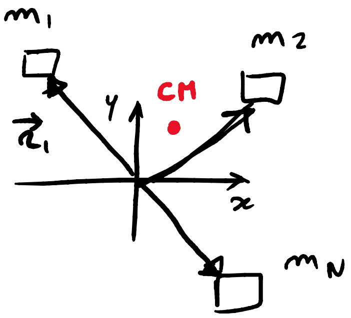
center of mass CM





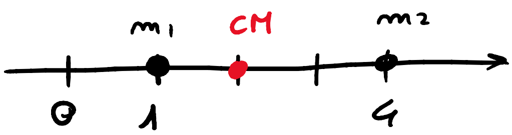
$$\vec{r}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_N \vec{r}_N}{m_1 + m_2 + \dots + m_N}$$

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \dots m_N x_N}{m_1 + m_2 + \dots + m_N}$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + \dots + m_N y_N}{m_1 + m_2 + \dots + m_N}$$

$m_1 = 4\ kg \qquad x_1 = 1\ m$

$m_2 = 2\ kg \qquad x_2 = 4\ m$



$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{4\,kg \cdot 1\,m + 2\,kg \cdot 4\,m}{4\,kg + 2\,kg} =$$

$$= \frac{12\,\cancel{kg} \cdot m}{6\,\cancel{kg}} = 2\ m$$

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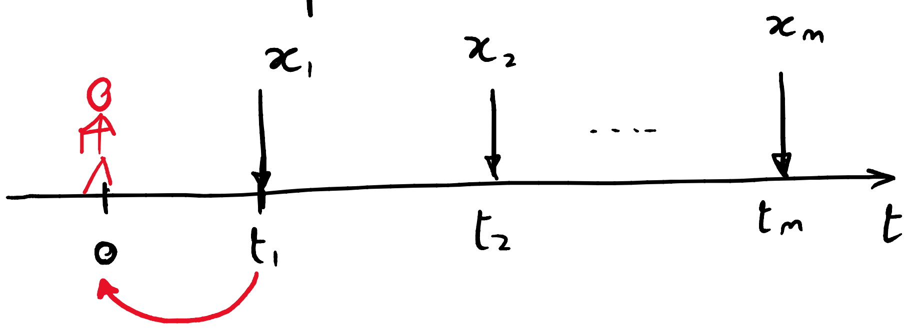
DURATION

$$CF = \{(x_1, x_2, \dots, x_m); (t_1, t_2, \dots, t_m)\}$$

we assume $x_i \geq 0 \quad \forall\ i \in \{1, \dots, m\}$

The duration is the weighted average of the time of the cash flow where the weights are the present values of the cash flows.



V_i : the present value of investment x_i

$$V_1 = x_1 (1 + i(0, t_1))^{-t_1}$$

$$V_2 = x_2 (1 + i(0, t_2))^{-t_2}$$

$$\vdots$$

$$V_m = x_2 (1 + i(0, t_m))^{-t_m}$$

$$D = \frac{x_1(1+i(0,t_1))^{-t_1} t_1 + x_2(1+i(0,t_2))^{-t_2} t_2 + \dots + x_m(1+i(0,t_m))^{-t_m} t_m}{x_1(1+i(0,t_1))^{-t_1} + x_2(1+i(0,t_2))^{-t_2} + \dots + x_m(1+i(0,t_m))^{-t_m}}$$

EX32: Consider the following cash flow

$$OF1 = \{(1, 1, 1, 100); (1, 2, 3, 4)\}$$

and the following term structure of spot interest rates

$$i(0,1) = 0.07,\ i(0,2) = 0.075,\ i(0,3) = 0.065,\ i(0,4) = 0.08.$$

In order to calculate the duration

$$D = \frac{V_1 t_1 + V_2 t_2 + V_3 t_3 + V_4 t_4}{V_0}$$

$$V_0 = V_1 + V_2 + V_3 + V_4$$

$$V_1 = x_1 (1 + i(0,1))^{-t_1} \cong 0.9346$$

$$V_2 = x_2 (1 + i(0,2))^{-t_2} \cong 0.8653$$

$$V_3 = x_3 (1 + i(0,3))^{-t_3} \cong 0.8278$$

$$V_4 = x_4 (1 + i(0,4))^{-t_4} \cong 73.5030$$

$$V_0 = V_1 + V_2 + V_3 + V_4 = 76.1307$$

$$D = \frac{V_1 t_1 + \dots + V_4 t_4}{V_0} \; \frac{\cancel{€} \cdot \text{year}}{\cancel{€}} = 3.9296 \text{ year}$$

If the interest rate is constant:

$$i(0, t_k) = i \quad \forall \; k \in \{1, \dots, m\}$$

In this case we talk about Macauly duration

$$x_1(1+i)^{-t_1} t_1 + x_2(1+i)^{-t_2} t_2 + \dots + x_m(1+i)^{-t_m} t_m$$

$$D_{MAC} = \frac{x_1(1+i)^{-t_1} t_1 + x_2(1+i)^{-t_2} t_2 + \dots + x_m(1+i)^{-t_m} t_m}{x_1(1+i)^{-t_1} + x_2(1+i)^{-t_2} + \dots + x_m(1+i)^{-t_m}}$$

t	x_t C_t
1	6
2	6
3	6
4	106

$i = 0.055$

SENSITIVITY : $\dfrac{V_0'}{V_0} = \dfrac{\frac{dV_0}{di}}{V_0}$

assumption 1 :
i : it is assumed to be constant

assumption 2 : $t_1 = 1 \quad t_2 = 2 \quad t_3 = 3 \quad \dots \quad t_m = m$

Under these circumstances, the Macaulay duration

$$D_{MAC} = \frac{x_1(1+i)^{-t_1} t_1 + x_2(1+i)^{-t_2} t_2 + \dots + x_m(1+i)^{-t_m} t_m}{x_1(1+i)^{-t_1} + x_2(1+i)^{-t_2} + \dots + x_m(1+i)^{-t_m}}$$

becomes

$$D_{MAC} = \frac{\overbrace{x_1(1+i)^{-1} + 2x_2(1+i)^{-2} + \dots + m x_m(1+i)^{-m}}^{(*)}}{\underbrace{x_1(1+i)^{-1} + x_2(1+i)^{-2} + \dots + x_m(1+i)^{-m}}_{V_0}}$$

$$(*) = D_{MAC} \cdot V_0$$

$$V_0 = x_1(1+i)^{-1} + x_2(1+i)^{-2} + \dots + x_m(1+i)^{-m}$$

$$V_0' = \frac{dV_0}{di} = -x_1(1+i)^{-2} - 2x_2(1+i)^{-3} - \dots - m x_m (1+i)^{-m-1} =$$

$$= -(1+i)^{-1} \underbrace{\left[x_1 (1+i)^{-1} + 2x_2(1+i)^{-2} + \dots + m x_m (1+i)^{-m} \right]}_{*}$$

$$* = -(1+i) V_0'$$

$$\frac{D_{MAC} \cdot \cancel{V_0}}{-(1+i)\cancel{V_0}} = \frac{\cancel{-}\cancel{(1+i)}\, V_0'}{\cancel{-}\cancel{(1+i)}\, V_0}$$

$$\underbrace{\frac{V_0'}{V_0}}_{\text{sensitivity}} = \underbrace{-\frac{1}{1+i} D_{MAC}}_{\text{modified duration}}$$

EX34: from EX31 we can now calculate the modified duration:

$$D_{MAC} = 3.6761$$

$$D_{MOD} = -\frac{1}{1+0.055} \cdot 3.6761 \cong -3.4845$$