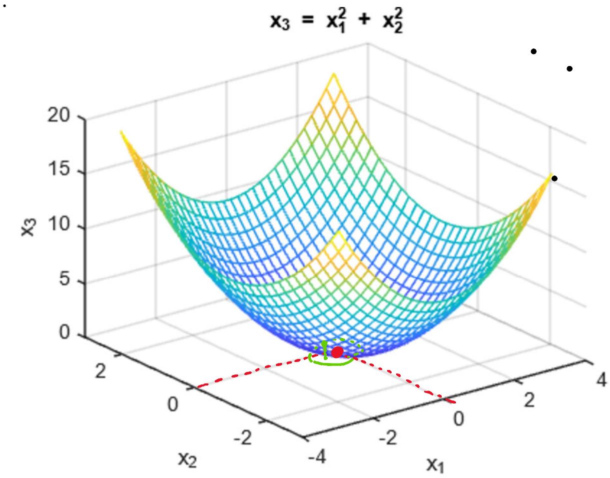
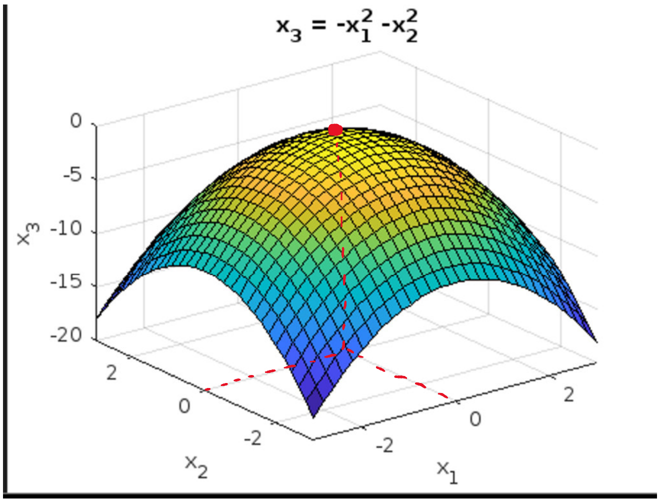
QUADRATIC FORMS

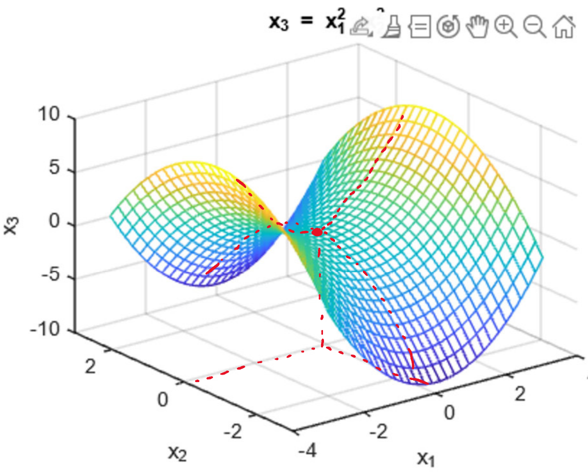
martedì 21 marzo 2023 08:41



The origin $(0,0)$ is a strict global minimum

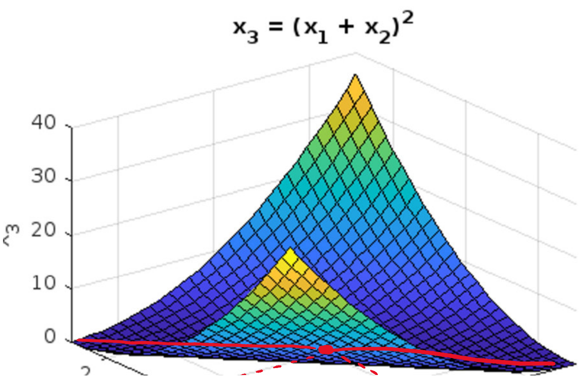


The origin is a strict global maximum

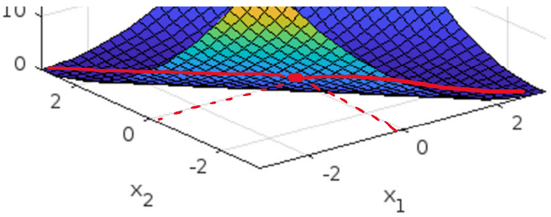


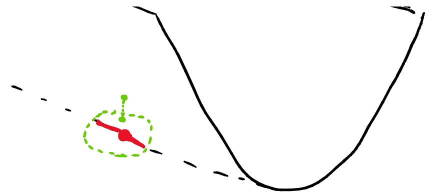
The origin $(0,0)$ is a saddle point

$$z = x^2 - y^2$$



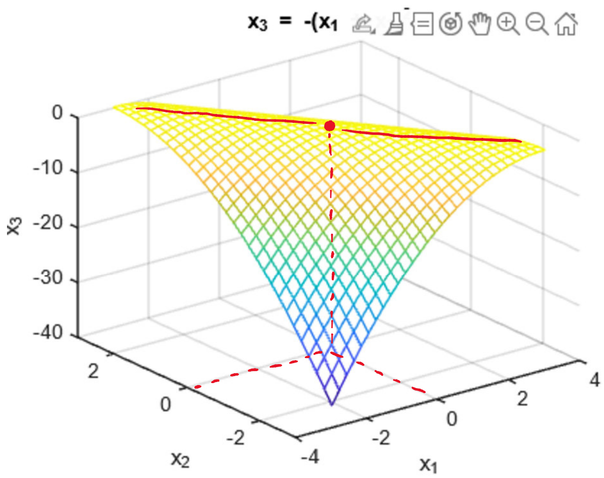
The origin $(0,0)$ is a global minimum but it is not strict





$$z = (x+y)^2 =$$

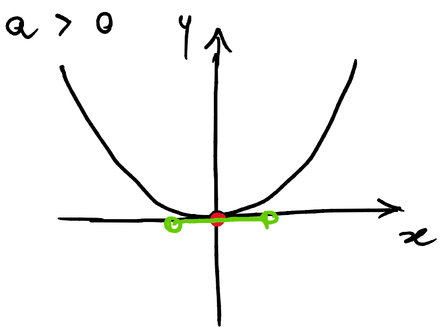
$$= x^2 + \underbrace{2xy}_{\text{degree } = 2} + y^2$$

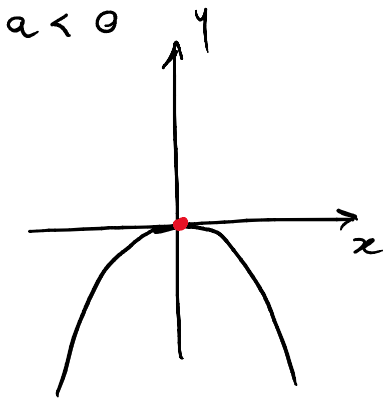


The origin $(0,0)$ is a global maximum but it is not strict

$y = f(x)$

$y = ax^2$





if $a > 0$, $(0,0)$ global minimum

if $a < 0$, $(0,0)$ global maximum

Eigenvalues

$Q(x) = ax^2$ quadratic form from $\mathbb{R}$ to $\mathbb{R}$

$Q(x_1, x_2) = a_{11} x_1^2 + a_{12} x_1 x_2 + a_{22} x_2^2$ ←
quadratic form from $\mathbb{R}^2$ to $\mathbb{R}$

In the initial example, we had

$Q(x_1, x_2) = x_1^2 + x_2^2$

$a_{11} = 1, \quad a_{12} = 0, \quad a_{22} = 1$

Ex: if $a_{11} = 2, \quad a_{12} = 3, \quad a_{22} = 4$

$2x_1^2 + 3x_1 x_2 + 4x_2^2$

$Q(x_1, x_2, x_3) = a_{11} x_1^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3 +$
$+ a_{22} x_2^2 + a_{23} x_2 x_3 + a_{33} x_3^2$

This is a quadratic form from $\mathbb{R}^3$ to $\mathbb{R}$

m M

A quadratic form from $\mathbb{R}^n$ to $\mathbb{R}$ n N

$Q(x_1, x_2, \dots, x_n)$

$Q(x_1, x_2) = a_{11} x_1^2 + a_{12} x_1 x_2 + a_{22} x_2^2$

Multiplication of a row vector by a column vector.

It is possible to do the multiplication only if the two vectors have the same length

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad \text{not possible}$$

$$\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 4 + (-1) \cdot 5 =$$

$$= 3 + 8 - 5 = 6$$

Multiplication of a matrix by a vector

The number of columns of the matrix has to be equal to the number of rows of the vector

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{NOT POSSIBLE}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2(-2) \\ 3 \cdot 1 + 4(-2) \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

Multiplication of a matrix by a matrix

$A \quad B$

The multiplication is defined ~~only~~ ONLY when the number of columns of A is equal to the number of rows of B.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{bmatrix}$$

NOT POSSIBLE

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}} = \begin{bmatrix} [1\ 2]\begin{bmatrix}1\\1\end{bmatrix} & [1\ 2]\begin{bmatrix}1\\-1\end{bmatrix} & [1\ 2]\begin{bmatrix}0\\1\end{bmatrix} \\ [3\ 4]\begin{bmatrix}1\\1\end{bmatrix} & [3\ 4]\begin{bmatrix}1\\-1\end{bmatrix} & [3\ 4]\begin{bmatrix}0\\1\end{bmatrix} \end{bmatrix} =$$

2×2 $\qquad$ 2×3

$$= \begin{bmatrix} 3 & -1 & 2 \\ 7 & -1 & 4 \end{bmatrix}$$

2×2

$$\begin{array}{cc} & k \\ m & \begin{bmatrix} \quad & \quad \end{bmatrix} \end{array} \begin{array}{cc} n & \\ \begin{bmatrix} \quad & \quad \end{bmatrix} & k \end{array} = \begin{array}{cc} & n \\ m & \begin{bmatrix} \quad & \quad \end{bmatrix} \end{array}$$

$\qquad A \qquad\qquad B$

$x_1^2 + x_2^2 + 2x_1x_2$ $\qquad$ ✓

$x_1^2 + x_2^2 + 2x_1$ $\qquad$ ✗

degree 1

$x_1^2 + 2x_2^2 - 5$ $\qquad$ ✗

degree 0

$-5x_2^0$

$$x_1 x_2 x_3 - 2x_1^2 + 3x_1x_3$$

$\uparrow$
degree 3

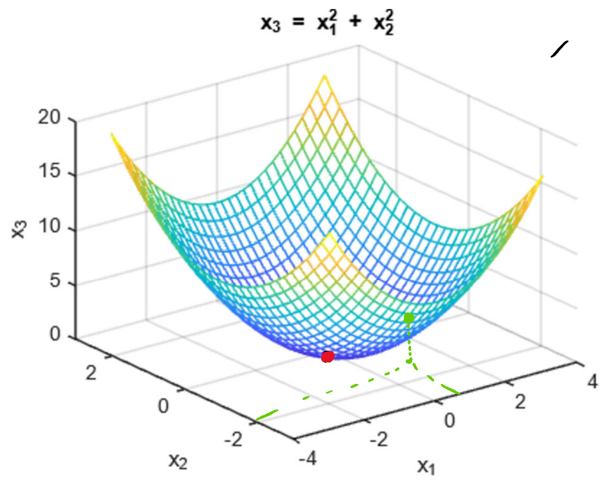
$$Q(x_1, \dots, x_m) = \sum_{i,j} a_{ij} x_i x_j \leftarrow$$

quadratic form from $\mathbb{R}^m$ to $\mathbb{R}$

In the slides:

$$Q(x_1, \dots, x_m) = a_{11}x_1^2 + a_{22}x_2^2 + \dots + a_{mm}x^m +$$

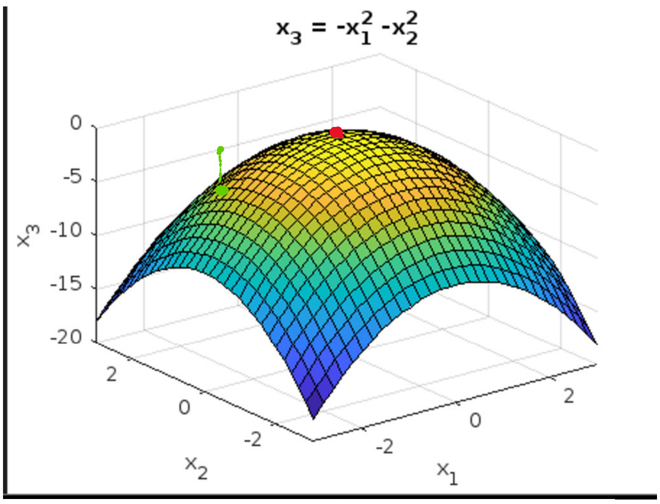
$$+ a_{12}x_{12} + \dots + a_{m1}x_m x_1 + \dots$$



The quadratic form $Q(x,y)$ or $Q(x_1, x_2)$ is said to be positive definite

$$Q(x_1, x_2) > 0$$

$$\forall (x_1, x_2) \neq (0,0)$$



The quadratic form is said to be negative definite

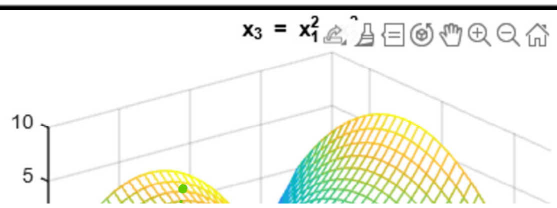
$$Q(x_1, x_2) < 0$$

$$\forall (x_1, x_2) \neq (0,0)$$

$\forall$: for each, for all

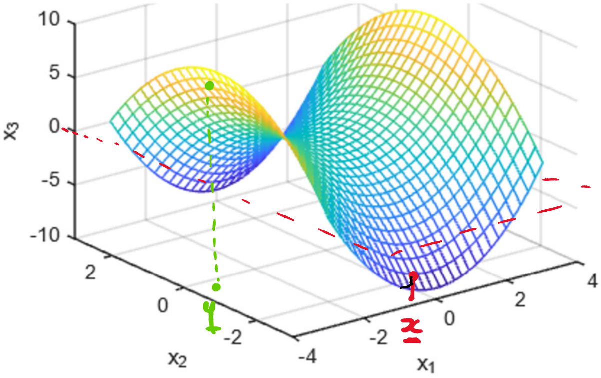
$\exists$: exists, there exists

| or : such that



In this case the quadratic form is indefinite

$\exists$

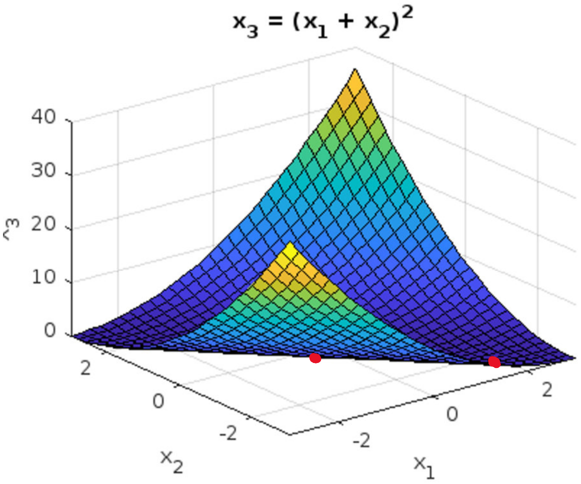


form is indefinite

$\exists \; \underline{x}, \underline{y} \;|$

$Q(\underline{x}) < 0$ and

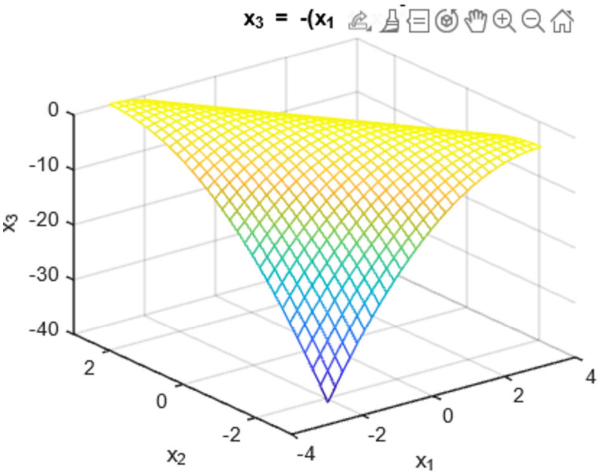
$Q(\underline{y}) > 0$



In this case the quadratic form is positive semi definite

$Q(x_1, x_2) \geq 0$

$\forall \, (x_1, x_2) \neq (0, 0)$



Negative semidefinite

$Q(x_1, x_2) \leq 0$

$\forall \, (x_1, x_2) \neq 0$

$$Q(x_1, x_2) = 2x_1^2 + 4x_1x_2 + 3x_2^2$$

$$[x_1 \;\; x_2] \underbrace{\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= \left(\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2x_1 & 4x_1 + 3x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= 2x_1 \cdot x_1 + (4x_1 + 3x_2)\, x_2 =$$

$$= 2x_1^2 + 4x_1x_2 + 3x_2^2$$

Therefore, we can always associate a matrix A to a quadratic form

In order to study the nature of the quadratic form (if is positive definite, or negative definite), it will be enough to study the properties of the associated matrix.

$$Q(x_1, x_2) = 2x_1^2 + 4x_1x_2 + 3x_2^2 =$$

$$= 2x_1^2 + 2x_1x_2 + 2x_1x_2 + 3x_2^2 =$$

$$= 2x_1^2 + 2x_1x_2 + 2x_2x_1 + 3x_2^2 =$$

$$= \underbrace{\begin{bmatrix} x_1 & x_2 \end{bmatrix}}_{\underline{x}^T} \underbrace{\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So there is not a single matrix such that we can express the quadratic form.

such that we can express ' the quadratic form.
We choose a matrix that is symmetric that is a matrix that it is equal to its transpose

$$A = A^T$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$ this is not a symmetric matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$ this is symmetric

So I can alway write a quadratic form from $\mathbb{R}^n$ to $\mathbb{R}$ as

$$Q(\underline{x}) = \underline{x}^T \underline{\underline{A}}\, \underline{x}$$ with $\underline{\underline{A}}$ symmetric.

Let $\underline{\underline{A}}$ be a symmetric matrix. Then $\underline{\underline{A}}$ is:

- positive definite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} > 0 \quad \forall\, \underline{x} \neq \underline{0}$
- negative definite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} < 0 \quad \forall\, \underline{x} \neq 0$
- positive semidefinite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} \geq 0 \quad \forall\, \underline{x} \neq \underline{0}$
- negative semidefinite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} \leq 0 \quad \forall\, \underline{x} \neq \underline{0}$
- indefinite: if $\exists\, \underline{x}, \underline{y} \neq \underline{0} \mid \underline{x}^T \underline{\underline{A}}\, \underline{x} > 0$ and $\underline{y}^T \underline{\underline{A}}\, \underline{y} < 0$