$Q(\underline{x}) = \sum_{i,j} a_{ij} x_i x_j$ or

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$Q(\underline{x}) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} x_i x_j$ quadratic form

It can always be expressed as

$Q(\underline{x}) = \underline{x}^T \underline{\underline{A}}\, \underline{x}$ with $\underline{\underline{A}}$ symmetric.

Let $\underline{\underline{A}}$ be a symmetric matrix. Then $\underline{\underline{A}}$ is:

- positive definite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} > 0 \quad \forall\, \underline{x} \neq \underline{0}$
- negative definite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} < 0 \quad \forall\, \underline{x} \neq 0$
- positive semidefinite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} \geq 0 \quad \forall\, \underline{x} \neq \underline{0}$
- negative semidefinite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} \leq 0 \quad \forall\, \underline{x} \neq \underline{0}$
- indefinite: if $\exists\ \underline{x}, \underline{y} \neq \underline{0} \mid \underline{x}^T \underline{\underline{A}}\, \underline{x} > 0$ and $\underline{y}^T \underline{\underline{A}}\, \underline{y} < 0$

EX 19: *consider the quadratic form*

$$Q = \underbrace{2x_1^2 + 3x_2^2 - x_3^2}_{\text{remain the same}} + \underbrace{4x_1x_3 - 6x_2x_3}_{\text{I take half of them}} =$$

$$= 2x_1^2 + 3x_2^2 - x_3^2 + 2x_1x_3 + \underbrace{2x_1x_3}_{\text{I change the order of the variables}} - 3x_2x_3 \underbrace{- 3x_2x_3}_{} =$$

$$= 2x_1^2 + 3x_2^2 - x_3^2 + 2x_1x_3 + 2x_3x_1 - 3x_2x_3 - 3x_3x_2 =$$

$2x_1 x_1$ rows columns $3x_2x_2$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{array}{c} \\ 1 \\ \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & -3 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

rows columns

$$= [x_1 \quad x_2 \quad x_3] \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\underline{x}^T \quad \underline{\underline{A}} \quad \underline{x}$$

While if we have the following matrix

$$B = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$Q(\underline{x}) = \underline{x}^T \underline{\underline{B}} \underline{x} = [x_1 \quad x_2 \quad x_3] \underbrace{\begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} =$$

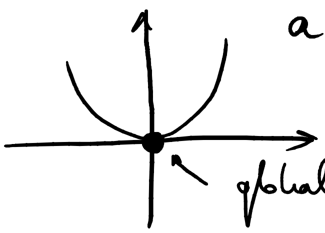
$$= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} -2x_1 + x_2 + 2x_3 \\ x_1 \\ 2x_1 + x_3 \end{bmatrix} =$$

$$= x_1(-2x_1 + x_2 + 2x_3) + x_2 x_1 + x_3(2x_1 + x_3) =$$

$$= -2x_1^2 + \underline{x_1 x_2} + \underline{\underline{2x_1 x_3}} + \underline{x_2 x_1} + \underline{\underline{2x_1 x_3}} + x_3^2 =$$

$$= -2x_1^2 + x_3^2 + 2x_1 x_2 + 4x_1 x_3$$

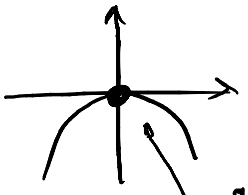
$$y = ax^2$$



$a > 0$

positive definite

global strict minimum

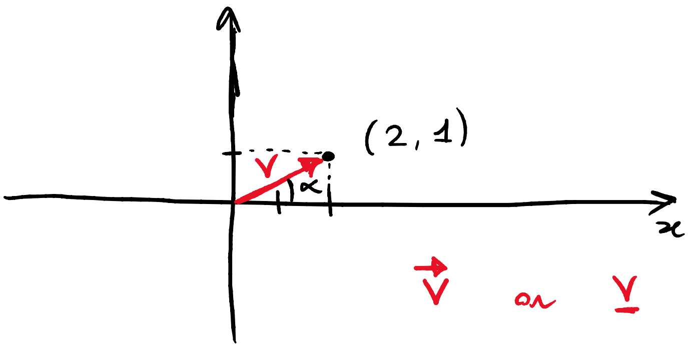


$a < 0$

negative definite

global strict maximum

To state whether a quadratic form is positive (semi) definite or negative (semi) definite or indefinite, we look at the eigenvalues of the symmetric matrix $\underline{\underline{A}}$ associated to the quadratic form.

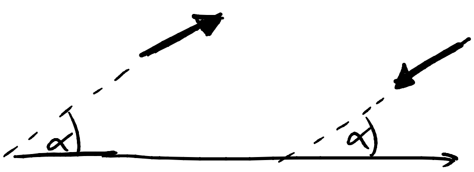


$\vec{v}$ or $\underline{v}$ or $\mathbf{v}$ (in bold) v

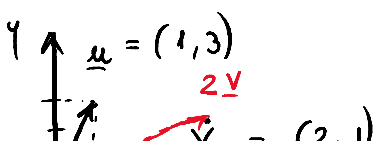
$$\underline{v} = (2, 1)$$

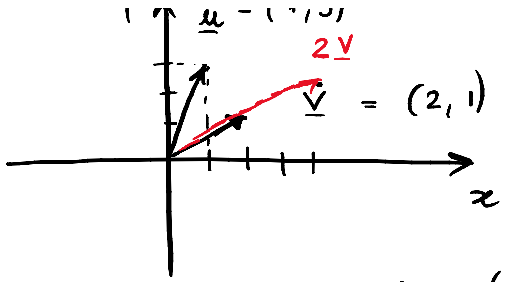
A vector is defined with three features:

- its length (or module) (how long is the arrow)
- its direction (angle α formed with the x axis)
- its verse (where the arrow points)



- same module
- same direction
- opposite verse





$2\underline{v} = (4,2)$ it is parallel to the initial vector $\underline{v}$

$$\boxed{\text{If two vectors are parallel then one of them is obtained multiplying the other one by a number}}$$

Put it in other words. if I have two vectors $\underline{x}$ and $\underline{y}$ and

$\underline{y} = \alpha \underline{x}$, then $\underline{x}$ is parallel to $\underline{y}$.

EX 9: *consider the following matrices and vectors:*

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{y} = \underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 1 \\ 0\cdot 1 + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\,\underline{x}$$

so the vectors $\underline{x}$ and $\underline{y}$ are parallel.

$$\underline{z} = \underline{B}\,\underline{x} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 + 2\cdot 1 \\ 1\cdot 1 - 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} =$$

$$= 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so the vectors $\underline{x}$ and $\underline{z}$ are not parallel so the vectors $\underline{x}$ and $\underline{z}$ are not parallel

Let $\underline{\underline{A}}$ be a square matrix, we are looking for a number λ such that $\underline{\underline{A}}\,\underline{x}$ is parallel to $\underline{x}$, that means

$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x}$$

Such a number is called eigenvalue of the matrix $\underline{\underline{A}}$.

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$\underline{\underline{A}}\,\underline{x} - \lambda\,\underline{x} = \underline{0}$ ← this is a vector

In the previous example we found that:

$$\underline{\underline{A}}\,\underline{x} = 3\,\underline{x}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $\underline{\underline{A}}\,\underline{x} - \lambda\,\underline{x} = \underline{0}$ ← so this is a vector and not a simple number

$\underline{\underline{I}}$: identity matrix $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ & & \ddots & 0 \\ 0 & 0 & & 1 \end{bmatrix}$

$$5 \cdot 1 = 5$$

$$1 \cdot 5 = 5$$

$$1 \cdot 5 = 5$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 0 & 1\cdot 0 + 2\cdot 1 \\ 0\cdot 1 + 3\cdot 0 & 0\cdot 0 + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 0\cdot 3 \\ 0\cdot 2 + 1\cdot 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So, writing $\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x}$ is the same as

$$\underline{\underline{A}}\,\underline{x} - \lambda\,\underline{x} = \underline{0}$$

$$\underline{\underline{A}}\,\underline{x} - \lambda\,\underline{\underline{I}}\,\underline{x} = \underline{0}$$

$$(\underline{\underline{A}} - \lambda\,\underline{\underline{I}})\,\underline{x} = \underline{0}$$

So, if I want to compute the eigenvalues of the matrix $\underline{\underline{A}}$, I need to solve the linear system (homogeneous)

$$(\underline{\underline{A}} - \lambda\,\underline{\underline{I}})\,\underline{x} = \underline{0}$$

RHS

$$\begin{cases} \overset{0}{x_1} - \overset{0}{x_2} = 0 \\ \underset{0}{x_2} + 2\underset{0}{x_3} = 0 \end{cases}$$

Homogeneous means that the right hand side (RHS) is zero.

Linear homogeneous systems have always the zero solution (i.e., all the variables are set to 0)

Example 1:

$$\begin{cases} x + y = 0 \\ x - y = 0 \end{cases} \qquad \begin{cases} y + y = 0 \\ x = y \end{cases} \qquad \begin{cases} 2y = 0 \\ x = y \end{cases}$$

$$\begin{cases} y = 0 \\ x = 0 \end{cases}$$ in this case, this is the only solution.

Example 2:

$$\begin{cases} x + y - z = 0 \\ x - y = 0 \end{cases} \qquad \begin{cases} y + y - z = 0 \\ x = y \end{cases}$$

$$\begin{cases} 2y = z \\ x = y \end{cases} \qquad \begin{cases} x = y \\ z = 2y \end{cases}$$ y is called a free variable

because it is free to assume any value

$\alpha \in \mathbb{R}$

$$\begin{cases} x = \alpha \\ y = \alpha \\ z = 2\alpha \end{cases}$$

$$S = \{ (\alpha, \alpha, 2\alpha) : \alpha \in \mathbb{R} \}$$

I have an infinite number of solutions.

If $\alpha = 1$ $\quad (1, 1, 2)$

If $\alpha = -2$ $\quad (-2, -2, -4)$

If $\alpha = 0$ $\quad (0, 0, 0)$ always present

With linear homogeneous systems $\underline{\underline{M}}\, \underline{x} = \underline{0}$ either we have:

1) only one solution $\quad \underline{x} = \underline{0}$

2)

1) only one solution $\underline{x} = \underline{0}$

2) an infinit number of solutions including $\underline{x} = \underline{0}$

I am in case 1) when $\det \underline{\underline{M}} \neq 0$

2) when $\det \underline{\underline{M}} = 0$

We impose $\det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) = 0$

$$\infty$$

4. $z = \sqrt{x^2 - 5xy + 4y^2}$

$$x^2 - 5xy + 4y^2 \geqslant 0$$

$$\Delta = 25y^2 - 16y^2 = 9y^2$$

$$x^2 - 5xy + 4y^2 = 0$$

$$x = \frac{5y \pm 3y}{2} \begin{cases} 4y \\ y \end{cases}$$

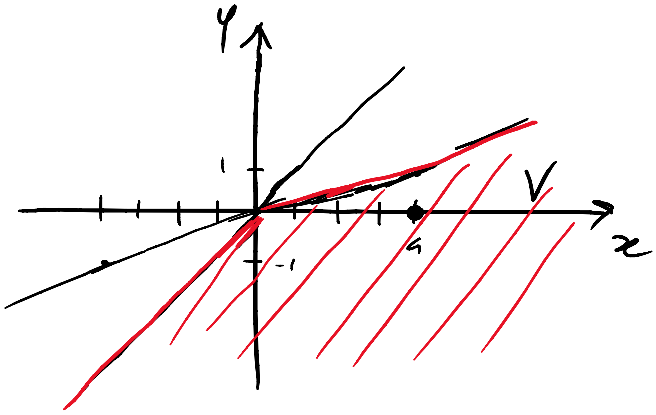
$$x^2 - 5xy + 4y^2 \geqslant 0$$

$$(x - 4y)(x - y) \geqslant 0$$

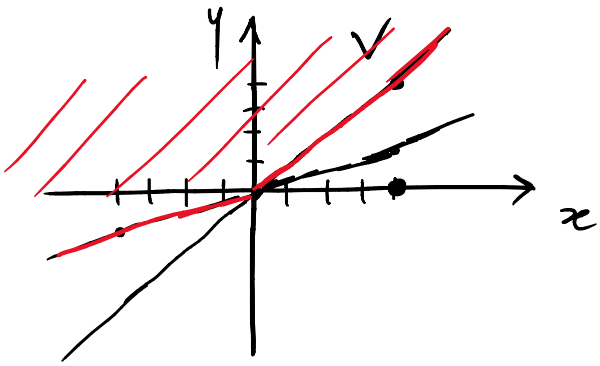
$$\begin{cases} x - 4y \geqslant 0 \\ x - y \geqslant 0 \end{cases} \vee \begin{cases} x - 4y < 0 \\ x - y < 0 \end{cases}$$

$$\begin{cases} x \geq 4y \\ x \geq y \end{cases}$$

$$\begin{cases} y \leq \frac{1}{4}x \leftarrow \quad 0 \leq 1 \\ y \leq x \end{cases}$$



$$\begin{cases} x - 4y < 0 \\ x - y < 0 \end{cases} \qquad \begin{cases} x < 4y \\ x < y \end{cases} \qquad \begin{cases} y > \frac{1}{4}x \\ y > x \end{cases}$$



$(4, 0)$

$0 > \frac{1}{4} \cdot 4$

$0 > 1$ NO