$$Q(\underline{x}) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\, x_i\, x_j = \underline{x}^T \underline{\underline{A}}\, \underline{x}$$

where $\underline{\underline{A}}$ is symmetric, i.e. $\underline{\underline{A}}^T = \underline{\underline{A}}$

$Q(\underline{x}) > 0 \quad \forall\ \underline{x} \neq 0$ positive definite

$\underline{x}^T \underline{\underline{A}}\, \underline{x} > 0$ $\quad \underline{\underline{A}}$ is posite definite

To know whether the matrix $\underline{\underline{A}}$ (and so the quadratic form is either

- positive (semi) definite $\quad \underline{\underline{A}} \succeq 0$ (positive definite $\succ$)
- negative (semi) definite $\quad \underline{\underline{A}} \preceq 0$
- indefinite

we need to study the sign of the eigenvalues of the matrix $\underline{\underline{A}}$

An eigenvalue is a scalar (a number) such that

$\underline{\underline{A}}\, \underline{x} = \lambda\, \underline{x}$ $\quad$ this means that the vector $\underline{\underline{A}}\, \underline{x}$ is paraller to vector $\underline{x}$

Such a vector is called eigenvector.

This is equivalent to solving the linear homogeneous system

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\, \underline{x} = \underline{0}$$

We don't want $\underline{x}$ to be $\underline{0}$ so we look for an infinite number of solutions.

Thus, we need to have $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) - 4 = 0$$

$$\cancel{4} - \lambda - 4\lambda + \lambda^2 - \cancel{4} = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\lambda(\lambda - 5) = 0 \qquad \lambda_1 = 0, \quad \lambda_2 = 5$$

Let's try to substitute $\lambda = 5$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}$$

$$(\underline{\underline{A}} - 5\underline{\underline{I}})\underline{x} = \underline{0}$$

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} \underline{x} = \underline{0}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \underline{x} = \underline{0} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4x_1 + 2x_2 \\ 2x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \cancel{-4} & \cancel{2} \\ 2 & -1 \end{bmatrix}$$

$$\begin{cases} -4x_1 + 2x_2 = 0 \\ 2x_1 - x_2 = 0 \end{cases}$$

$$\begin{cases} -4x_1 + 4x_1 = 0 \\ x_2 = 2x_1 \end{cases} \qquad \begin{cases} \cancel{0 = 0} \\ x_2 = 2x_1 \end{cases}$$

$$\begin{cases} x_1 = k \\ x_2 = 2k \end{cases}$$

$$S = \{ (k, 2k) \; : \; k \in \mathbb{R} \}$$

These solutions are the eigenvectors

For $k = 6$ we get $\underline{x} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$

$$\underline{\underline{A}}\, \underline{x} = \lambda \underline{x}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \cdot 6 + 2 \cdot 12 \\ 2 \cdot 6 + 4 \cdot 12 \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \end{bmatrix}$$

$$\lambda \underline{x} = 5 \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \end{bmatrix}$$

THEOREM: consider a quadratic form $Q(\underline{x}) = \underline{x}^T \underline{\underline{A}}\, \underline{x}$ with $\underline{\underline{A}}$ symmetric and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $\underline{\underline{A}}$. Then: $Q(\underline{x})$ is:

1) positive definite iif (if and only if) all the eigenvalues of $\underline{\underline{A}}$ are positive
2) negative definite iif all the eigenvalues of $\underline{\underline{A}}$ are negative

are negative

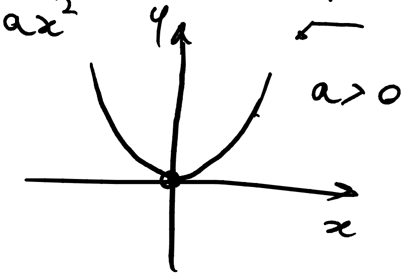
3) positive semidefinite iiff all the eigenvalues of $\underline{\underline{A}}$ are not negative and at least one is 0.

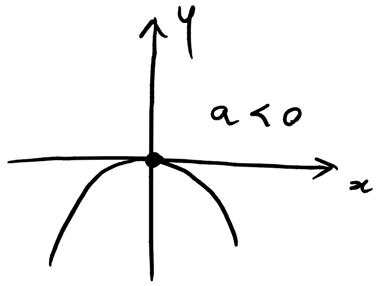
4) negative semidefinite iiff all the eigenvalues of $\underline{\underline{A}}$ are not positive and at least one is 0.

5) indefinite : iff $\underline{\underline{A}}$ admits both positive and negative eigenvalues.

This is a generalization of when we said

$y = ax^2$





$$Q(\underline{x}) = x_1^2 + x_2^2 - 4x_1x_2 =$$

$$= x_1^2 + x_2^2 - 2x_1x_2 - 2x_1x_2 =$$

$$= \underset{x_1 x_1}{1x_1^2} + \underset{x_2 x_2}{1x_2^2} - 2x_1x_2 - 2x_2x_1 =$$

$$= [x_1 \quad x_2] \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

$$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0$$

$$(\lambda - 1)^2 = 4$$

$$\lambda - 1 = \pm 2$$

$$\lambda = 1 \pm 2 \quad \begin{cases} \lambda_1 = 3 \\ \lambda_2 = -1 \end{cases}$$

so the quadratic form is indefinite

∞

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 6 & 4 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

This is a triangular matrix

For the quadratic forms, we require the matrix $\underline{\underline{A}}$ to be symmetric, but we can compute the eigenvalues of non-symmetric squared matrices.

$$\det \underline{\underline{A}} = ?$$

$$+1 \cdot \begin{vmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} =$$

$$= 1 \cdot 2 \, (-1) \cdot 3 = -6$$

A triangular matrix is a squared matrix

A triangular matrix is a squared matrix such that the elements below or above the main diagonal are all 0.

The determinant of a triangular matrix is given by the product of all the elements on the main diagonal.

Let us also compute the eigenvalues.

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 6 & 4 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & -1-\lambda & 1 \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(-1-\lambda)(3-\lambda) = 0$$

$$\lambda = 1 \vee \lambda = 2 \vee \lambda = -1 \vee \lambda = 3$$

So the eigenvalues of a triangular matrix are the values on the main diagonal

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

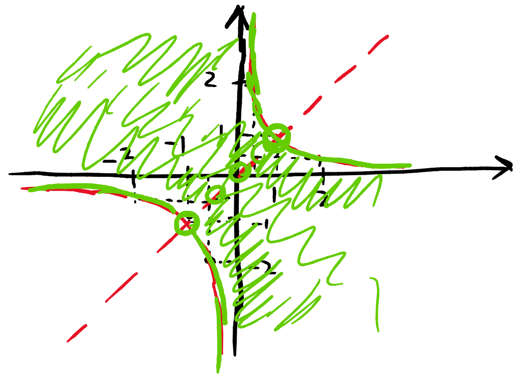
11. $z = \dfrac{1}{y-x} + \sqrt{1-xy}$

$$\begin{cases} y - x \neq 0 \\ 1 - xy \geq 0 \end{cases}$$

$1 - xy \geq 0$

$xy \leq 1$

$xy = 1$



$(0, 0)$

$0 \cdot 0 \leq 1$

$0 \leq 1$

$y - x \neq 0 \qquad y \neq x \qquad \underline{y = x}$

13. $z = \dfrac{\sin^{-1} \log_2 x + \sqrt{2x - y^2}}{\ln(x - y)}$

$\sin x \in [-1, 1]$

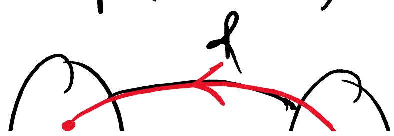
$\sin^{-1} x$

$$\begin{cases} x > 0 \\ x - y > 0 \\ \ln(x - y) \neq 0 \\ 2x - y^2 \geq 0 \\ -1 \leq \log_2 x \leq 1 \end{cases}$$

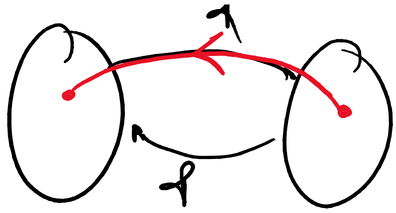
$$\begin{cases} x > 0 \\ x - y > 0 \\ \ln(x - y) \neq \ln 1 \\ 2x - y^2 \geq 0 \\ 2^{-1} \leq 2^{\log_2 x} \leq 2^1 \end{cases}$$

$$\begin{cases} x > 0 \\ x - y > 0 \\ x - y \neq 1 \end{cases}$$

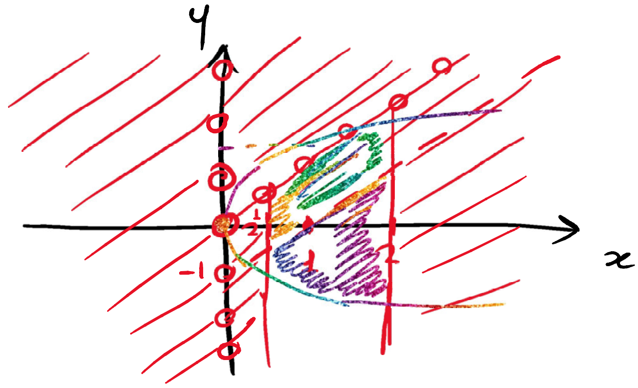
$f(f^{-1}(x)) = x$



$$\begin{cases} x - y \neq 1 \\ 2x - y^2 \geq 0 \\ \frac{1}{2} \leq x \leq 2 \end{cases}$$



$$\begin{cases} x > 0 \\ y < x \\ y \neq x - 1 \\ x \geq \frac{1}{2} y^2 \\ \frac{1}{2} \leq x \leq 2 \end{cases}$$



$$x = \frac{1}{2} y^2$$

$$x \geq \frac{1}{2} y^2 \qquad (1, 0)$$

$$1 \geq 0$$