mercoledì 29 marzo 2023 09:00

$f : A \subseteq \mathbb{R} \to \mathbb{R}$

$x_0 \in A$

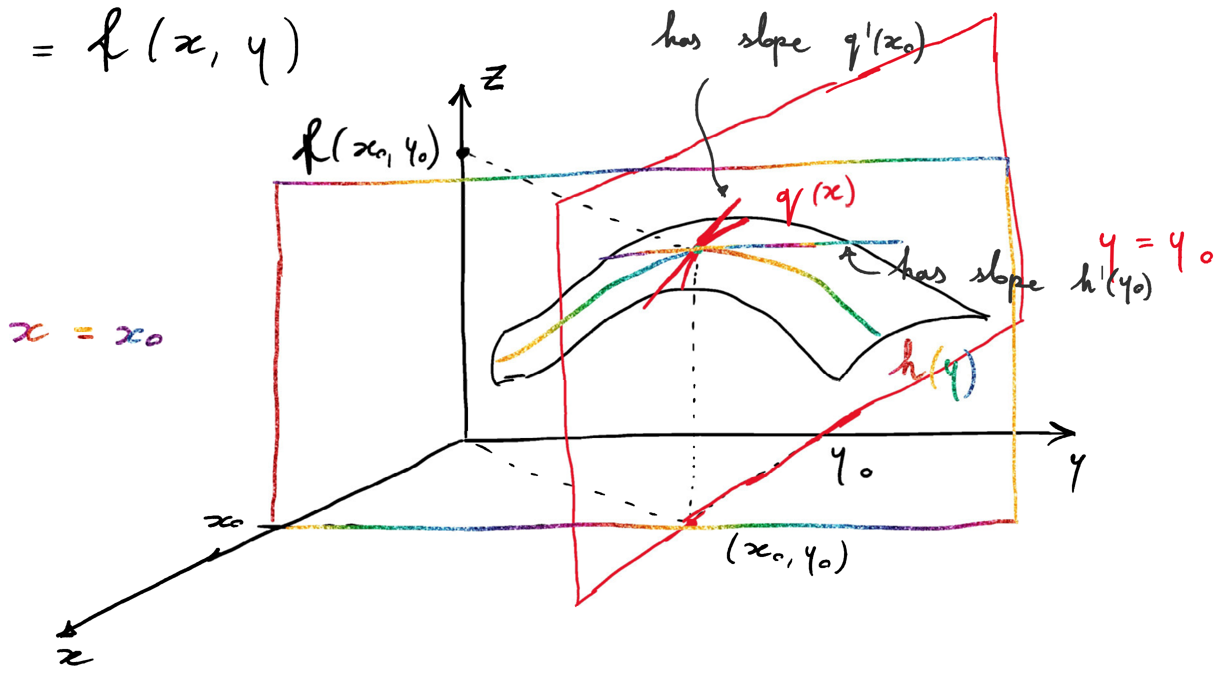
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

It does not alway exists.

$f : A \subseteq \mathbb{R}^m \to \mathbb{R}$

For example $m = 2$

$z = f(x, y)$



$$g(x) = f(x, y_0)$$

$$g'(x_0) = \lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} =: \frac{\partial f}{\partial x}(x_0, y_0) =$$

$$= f'_x(x_0, y_0)$$

if it exists

$$h(y) = f(x_0, y)$$

$$h'(y_0) = \lim_{k \to 0} \frac{h(y_0 + k) - h(y_0)}{k} =$$

$$= \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k} = f'_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$$

if it exists.

If the partial derivatives exist and are finite, we can collect them into a vector

del or nabla

$$\nabla f(x_0, y_0) = \operatorname{grad} f(x_0, y_0) := \begin{bmatrix} f'_x(x_0, y_0) \\ f'_y(x_0, y_0) \end{bmatrix}$$

If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right]^T$$

∞

$$z = x^2 + 2y^2 + 5xy$$

- compute the gradient in general
- compute the gradient at point $(1, 2)$

$$f'_x = 2x + 5y \qquad f'_y = 4y + 5x$$

$$\nabla f = \begin{bmatrix} 2x + 5y \\ 4y + 5x \end{bmatrix} \qquad \nabla f(1,2) = \begin{bmatrix} 2\cdot 1 + 5\cdot 2 \\ 4\cdot 2 + 5\cdot 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

$$f(x,y,z) = 2\sqrt{x} + 3xz + e^{y}$$

$$\operatorname{dom} f = \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0\}$$

Compute the gradient at point $(1, 0, 2)$

$$f(x,y,z) = 2\, x^{\frac{1}{2}} + 3xz + e^{y}$$

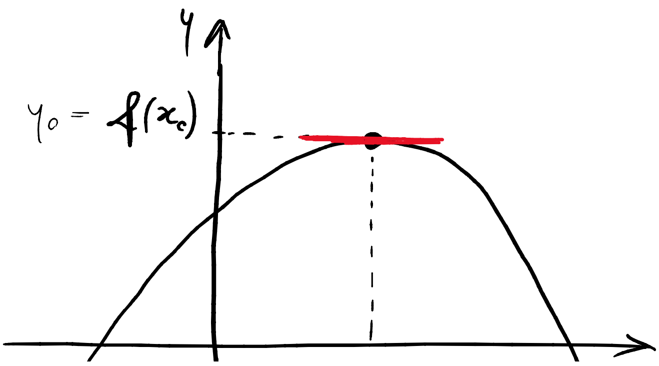
$$D\, x^{\alpha} = \alpha\, x^{\alpha - 1}$$

$$f'_x = 2 \cdot \frac{1}{2}\, x^{\frac{1}{2}-1} + 3z = x^{-\frac{1}{2}} + 3z = \frac{1}{\sqrt{x}} + 3z$$

$$f'_y = e^{y} \qquad f'_z = 3x$$

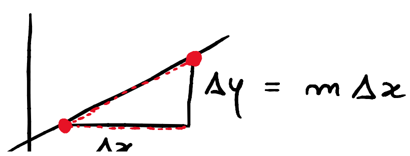
$$\nabla f = \left(\frac{1}{\sqrt{x}} + 3z,\ e^{y},\ 3x\right)^T$$

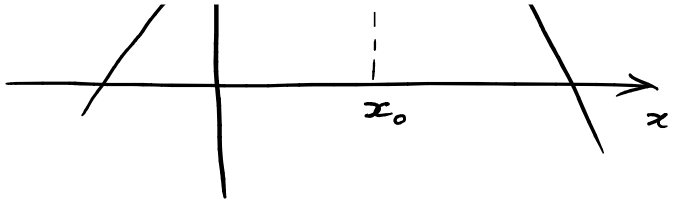
$$\nabla f(1,0,2) = \left(\frac{1}{\sqrt{1}} + 3\cdot 2,\ e^{0},\ 3\cdot 1\right)^T = (7,\ 1,\ 3)^T$$

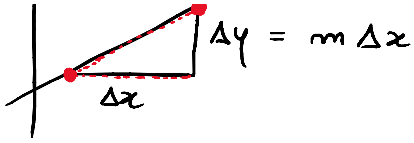


$$y = mx + q$$

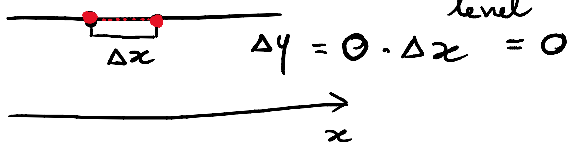
$\uparrow$ slope $\quad \uparrow$ intercept







Here, I remain at the same level



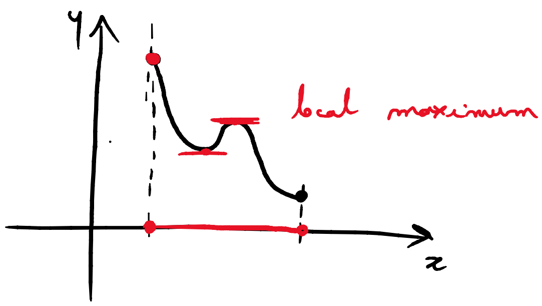
so the slope is 0

This happens for those lines that are parallel to the x axis

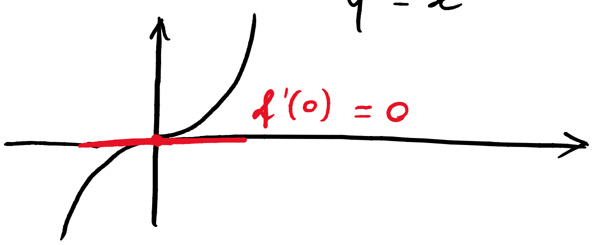
$$y = mx + q$$

If $m = 0$ $\qquad y = 0 \cdot x + q$

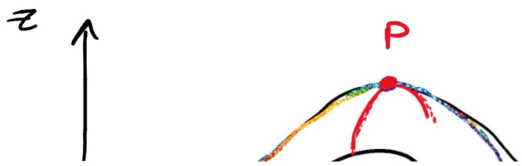
$y = q$ and this is the equation of a line parallel to the x axis.



$y = x^3$



Setting the derivative to 0 is a necessary condition.



$g(x) = f(x, y_0)$

$h(y) = f(x_0, y)$

$g(x) = f(x, y_0)$
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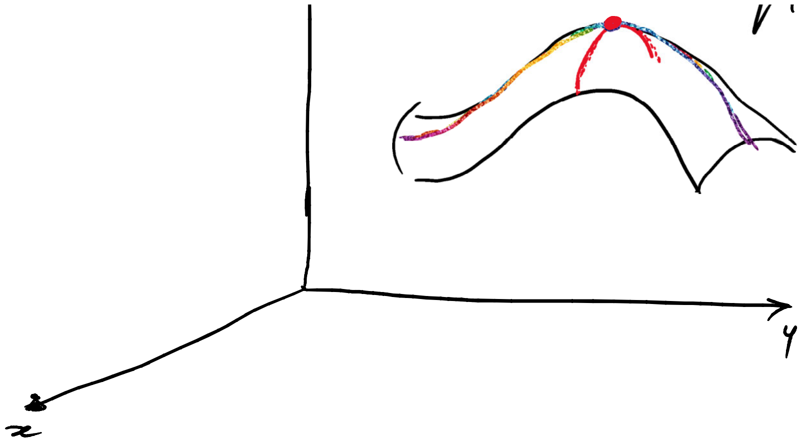


Point P is a maximum
But it is a maximum also for functions $g(x)$ and $h(y)$.

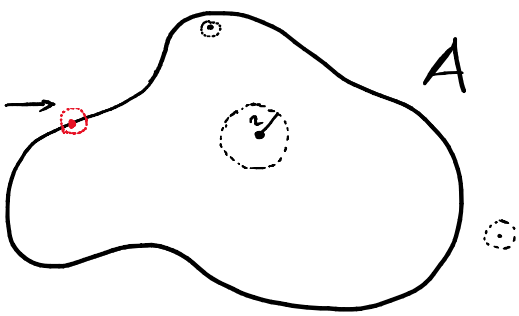
Therefore $g'(x_0) = 0$ and $h'(y_0) = 0$

So $f'_x(x_0, y_0) = 0$ and $f'_y(x_0, y_0) = 0$

So $\nabla f(x_0, y_0) = \underline{0}$

A point $\underline{x}^*$ is an interior point of A if there exists a whole r-ball about $\underline{x}^*$ in the domain A.

this is not an interior point



Function of class C^1

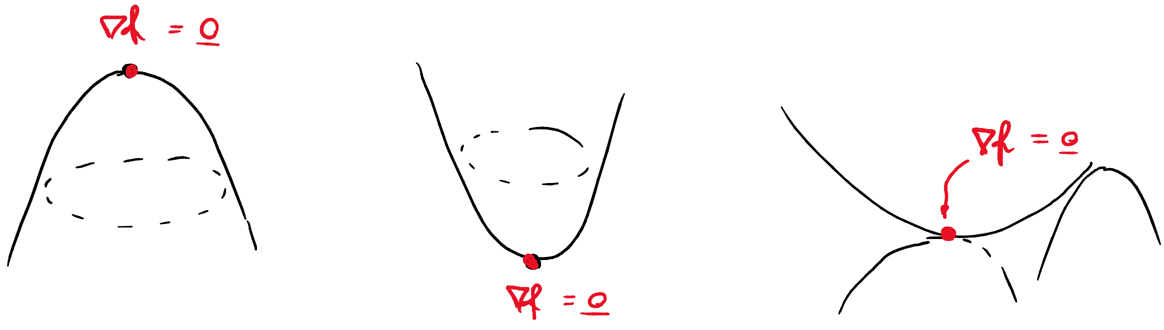
If all the partial derivatives of a

If all the partial derivatives of a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ are continuous in a point $\underline{x} \in A$ then f is said to be of class C^1 in $\underline{x}$.

FIRST ORDER CONDITION: let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and $\underline{x}^* \in A$ an interior point of A. If $\underline{x}^*$ is a local max or min. then $\frac{\partial f}{\partial x_i}(\underline{x}^*) = 0 \quad \forall i \in \{1, \dots, n\}$

$$\left(\text{or} \quad \nabla f(\underline{x}^*) = \underline{0} \right)$$

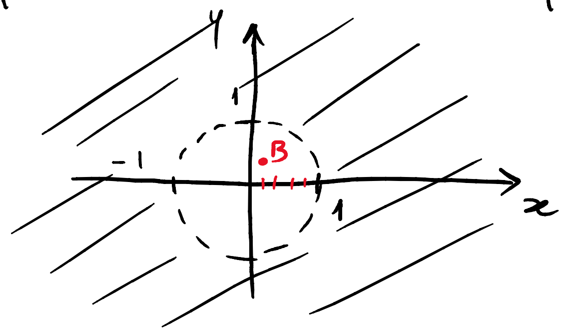
An interior point $\underline{x}$ is said to be a critical point if $\nabla f(\underline{x}) = \underline{0}$.



9. $f(x, y) = 2\ln(x^2 + y^2 - 1) + x + 2y$

$x^2 + y^2 - 1 > 0 \qquad x^2 + y^2 > 1^2$

$$x^2+y^2-1 > 0 \qquad x^2+y^2 > 1^2$$



$$\nabla f = \underline{0}$$

9. $f(x,y) = 2\ln(x^2+y^2-1) + x + 2y$

$$f'_x = 2 \frac{1}{x^2+y^2-1} \cdot 2x + 1 =$$

$$= \frac{4x}{x^2+y^2-1} + 1$$

$$f'_y = 2 \cdot \frac{1}{x^2+y^2-1} \cdot 2y + 2 =$$

$$= \frac{4y}{x^2+y^2-1} + 2$$

$$\nabla f(\underline{x}) = \underline{0}$$

$$\begin{cases} \dfrac{4x}{x^2+y^2-1} + 1 = 0 \\[2ex] \dfrac{4y}{x^2+y^2-1} + 2 = 0 \end{cases}$$

$$\begin{cases} \dfrac{4x}{x^2+y^2-1} = -1 \\[2ex] \dfrac{\cancel{4}^{2}y}{x^2+y^2-1} = -\cancel{2}^{1} \end{cases}$$

$$\begin{cases} \dfrac{\overset{2}{\cancel{4}}y}{x^2+y^2-1} = -\overset{1}{\cancel{2}} \end{cases}$$

$$\begin{cases} \dfrac{4x}{x^2+y^2-1} = -1 \\[2ex] (x^2+y^2-1)\,\dfrac{4x}{\cancel{x^2+y^2-1}} = \dfrac{2y}{\cancel{x^2+y^2-1}} \cdot (x^2+y^2-1) \end{cases}$$

provided $x^2+y^2-1 \neq 0$

it is always > 0 because it is the argument of the logarithm.

$$\begin{cases} \dfrac{4x}{x^2+y^2-1} = -1 \\ y = 2x \end{cases} \qquad \begin{cases} 4x = -x^2-y^2+1 \\ y = 2x \end{cases}$$

$$\begin{cases} 4x = -x^2-4x^2+1 \\ y = 2x \end{cases} \qquad \begin{cases} 5x^2+4x-1 = 0 \\ y = 2x \end{cases}$$

$$\begin{cases} x = \dfrac{-2 \pm 3}{5} \\ y = 2x \end{cases}$$

$$\begin{cases} x = -1 \\ y = -2 \end{cases} \quad \vee \quad \begin{cases} x = \dfrac{1}{5} \\ y = \dfrac{2}{5} \end{cases}$$

Two <u>candidates</u> maxima or minima:

$A(-1, -2)$ $\qquad$ $\cancel{B\left(\frac{1}{5}, \frac{2}{5}\right)}$

outside the domain