9. $f(x,y) = 2\ln(x^2+y^2-1) + x + 2y$

$$f'_x = \frac{4x}{x^2+y^2-1} + 1$$

$$f'_y = \frac{4y}{x^2+y^2-1} + 2$$

$A(-1, -2)$ candidate to be a min or max

$$\begin{tikzcd}
 & & f \arrow[dl, dash] \arrow[dr, dash] & & \\
 & f'_x \arrow[dl, dash] \arrow[d, dash] & & f'_y \arrow[d, dash] \arrow[dr, dash] & \\
f''_{xx} & f''_{xy} & & f''_{yx} & f''_{yy}
\end{tikzcd}$$

Second partial derivatives: if all the first partial derivatives are derivable again, then we can define:

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \underbrace{\frac{\partial^2 f}{\partial x^2}}_{\text{compact notation}} = f''_{xx}$$

example:

$$f'_x = \frac{4x}{x^2+y^2-1} + 1$$

$$f''_{xx} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{4x}{x^2+y^2-1} + 1\right) =$$

$$= \frac{4(x^2+y^2-1) - 4x \cdot 2x}{(x^2+y^2-1)^2} + 0 =$$

$$= \frac{4(x^2+y^2-1-2x^2)}{(x^2+y^2-1)^2} = \frac{4(-x^2+y^2-1)}{(x^2+y^2-1)^2}$$

$$\frac{\partial}{\partial y}\underbrace{\frac{\partial f}{\partial x}}_{f'_x} = \frac{\partial^2 f}{\partial y\,\partial x} = f''_{xy}$$

you read this as: "first you derive with respect to x and then with respect to y"

you read: "I derive with respect to y the first derivative with respect to x"

$$f'_x = \frac{4x}{x^2+y^2-1} + 1$$

$$f''_{xy} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}\left(\frac{4x}{x^2+y^2-1} + 1\right) =$$

$$= 4x\,\frac{\partial}{\ }\,(x^2+y^2\ \ 1)^{-1}$$

$$= 4x \, \frac{\partial}{\partial y} \left(x^2 + y^2 - 1\right)^{-1} =$$

$$= 4x \left[-1 \left(x^2 + y^2 - 1\right)^{-2} \cdot 2y \right] =$$

$$= - \frac{8xy}{\left(x^2 + y^2 - 1\right)^2}$$

$$f'_y = \frac{4y}{x^2 + y^2 - 1} + 2$$

$$f''_{yx} = \frac{\partial}{\partial x} \left(\frac{4y}{x^2 + y^2 - 1} + 2 \right) =$$

$$= 4y \, \frac{\partial}{\partial x} \left(x^2 + y^2 - 1\right)^{-1} =$$

$$= 4y \left[- \left(x^2 + y^2 - 1\right)^{-1-1} \cdot 2x \right] =$$

$$= - \frac{8xy}{\left(x^2 + y^2 - 1\right)^2}$$

$$f'_y = \frac{4y}{x^2 + y^2 - 1} + 2$$

$$f''_{yy} = \frac{\partial}{\partial y} \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[\frac{4y}{x^2 + y^2 - 1} + 2 \right] =$$

$$= \frac{4 \cdot \left(x^2 + y^2 - 1\right) - 4y \cdot 2y}{\left(x^2 + y^2 - 1\right)^2} =$$

$$= \frac{4 \cdot (x^2 + y^2 - 1) - 4y \cdot 2y}{(x^2 + y^2 - 1)^2} =$$

$$= \frac{4\,(x^2 + y^2 - 1 - 2y^2)}{(x^2 + y^2 - 1)^2} = \frac{4\,(x^2 - y^2 - 1)}{(x^2 + y^2 - 1)^2}$$

Function of class C^2: if all the second derivatives of f exist and are continuous, then f is said to be of class C^2

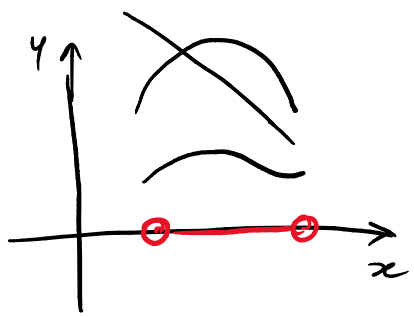
Schwartz theorem:

If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, A open set and $f \in C^2(A)$ then $\forall\, \underline{x} \in A$, $\forall\, i, j \in \{1, 2, \dots, n\}$

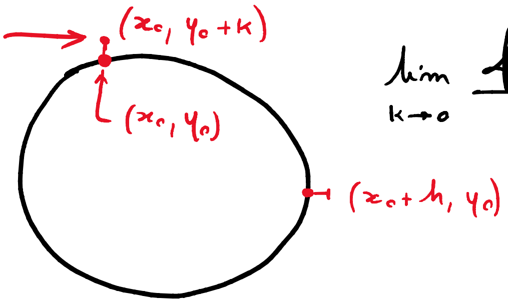
$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\underline{x}).$$

$\blacksquare$

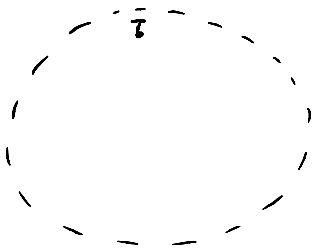
$C^2(A)$: the set of all the C^2 functions whose domain is A.



Here the function is not defined



$$\lim_{k\to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k}$$



Hessian matrix

$$\underline{\underline{H}}\, f = \begin{bmatrix} f''_{\underline{x_1}\, x_1} & f''_{\underline{x_1}\, x_2} & \cdots & f''_{\underline{x_1}\, x_m} \\ f''_{x_2\, x_1} & f''_{x_2\, x_2} & \cdots & f''_{x_2\, x_m} \\ \vdots & \vdots & \ddots & \vdots \\ f''_{x_m\, x_1} & f''_{x_m\, x_2} & \cdots & f''_{x_m\, x_m} \end{bmatrix} \longleftarrow$$

In our example, we have:

$$\underline{\underline{H}}\, f = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{4(-x^2+y^2-1)}{(x^2+y^2-1)^2} & -\dfrac{8xy}{(x^2+y^2-1)^2} \\ -\dfrac{8xy}{(x^2+y^2-1)^2} & \dfrac{4(x^2-y^2-1)}{(x^2+y^2-1)^2} \end{bmatrix}$$

$$\Big[\quad (x^2+y^2-1)^2 \qquad\qquad (x^2+y^2-1)^2 \quad \Big]$$

So, if $f \in C^2$, the Hessian matrix is symmetric

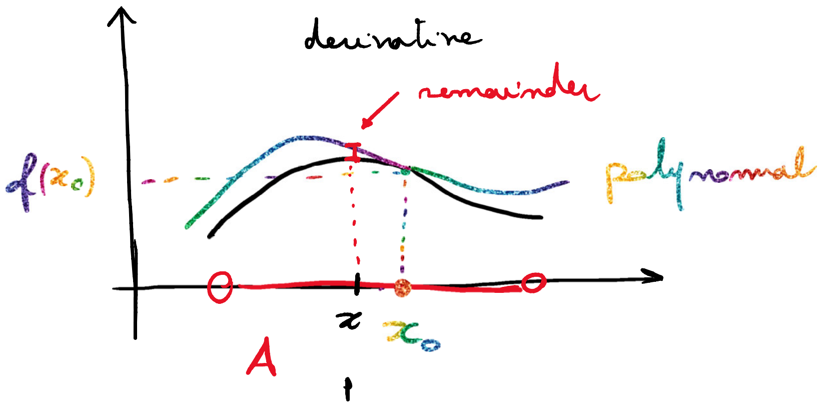
∞

TAYLOR POLYNOMIALS

Let $f : \underset{\text{open}}{A} \subseteq \mathbb{R} \to \mathbb{R}$ be a C^{k+1} function.
For any $x, x_0 \in A$ there exists a point c^* between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 +$$

$$+ \dots + \frac{1}{k!} \underbrace{f^{(k)}}_{k\text{-th order derivative}}(x_0) + \underbrace{\frac{1}{(k+1)!} f^{(k+1)}(c^*)(x-x_0)^{k+1}}_{\text{remainder } R_k(x,\, x_0)}$$



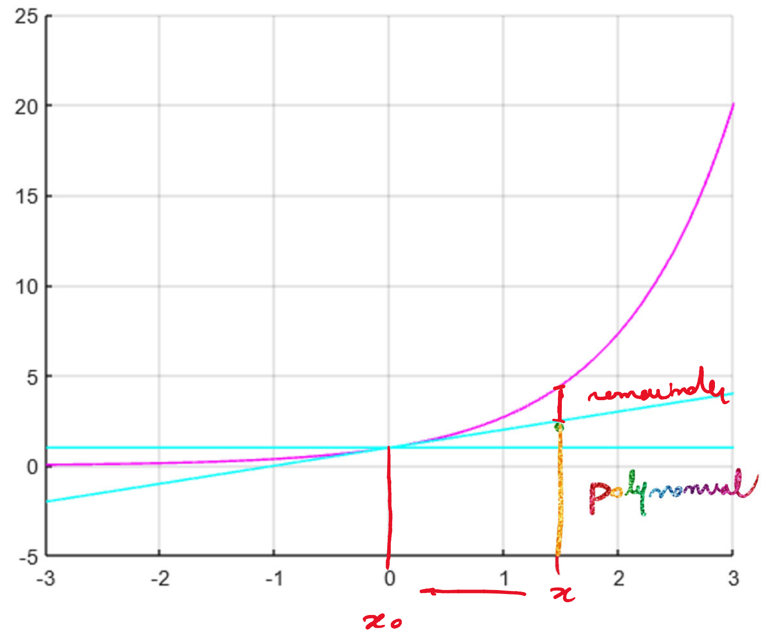
$$\lim_{x \to x_0} \frac{R_k(x, x_0)}{|x-x_0|^k} =$$

$$\cancel{\frac{x^5}{|x^4|}} \xrightarrow{x \to 0} 0$$

$$= \lim_{x \to x_0} \frac{1}{(k+1)!} f^{(k+1)}(c^*) \frac{(x-x_0)^{k+1}}{|x-x_0|^k} = 0$$

$$= \lim_{x \to x_0} \underbrace{\frac{1}{(k+1)!} f^{(k+1)}(c)}_{\text{constant}} \underbrace{\frac{(x-x_0)^{k+1}}{|x-x_0|^k}}_{\pm (x - x_0)} = 0$$

So if I stay in a neighborhood of x_0, the remainder tends to 0 and so substituting the function with the Taylor polynomial is a good approximation.



$$f : A \subseteq \mathbb{R}^n \to \mathbb{R} \quad , \quad f \in C^2(A)$$

$$f(\underline{x}) = f(\underline{x}_0) + \nabla f^T(\underline{x}_0) \underbrace{(\underline{x} - \underline{x}_0)} +$$

$$+ \frac{1}{2} (\underline{x} - \underline{x}_0)^T \underline{\underline{H}}_f(\underline{x}_0)(\underline{x} - \underline{x}_0) + R_2(\underline{x}, \underline{x}_0)$$

where
$$\lim_{\underline{x} \to \underline{x}_0} \frac{R_2(\underline{x}, \underline{x}_0)}{\| x - x_0 \|^2} = 0$$

where $\lim_{\underline{x} \to \underline{x}_0} \frac{[illegible]}{\|\underline{x} - \underline{x}_0\|^2} = 0$

norm

$$\|\underline{x} - \underline{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_m - y_m)^2} = d(\underline{x}, \underline{y})$$

$|x - x_0|$ the distance between x and x_0

$$f(x, y) = e^x \sin y$$

2-nd order Taylor polynomial in $(0, 0) = \underline{x}_0$

$$f(\underline{x}_0) = f(0, 0) = e^0 \cdot \sin 0 = 1 \cdot 0 = 0$$

$$f'_x = e^x \sin y \qquad f'_x(0, 0) = 0$$

$$f'_y = e^x \cos y \qquad f'_y(0, 0) = e^0 \cos 0 = 1$$

$$\nabla f(\underline{x}_0) = \nabla f(0, 0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$f''_{xx} = e^x \sin y \qquad f''_{xy} = e^x \cos y$$

$$f''_{yx} = e^x \cos y \qquad f''_{yy} = -e^x \sin y$$

$$\underline{\underline{H}}_f = \begin{bmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{bmatrix}$$

$$\underline{\underline{H}}_f(\underline{x}_0) = \underline{\underline{H}}_f(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$f(x) = f(x_0) + \nabla f(x_0)^T (x - x_0)$$
$(0,0)$

$$f(\underline{x}) = f(\underline{x}_0) + \nabla f(\underline{x}_0)^T (\underline{x} - \overset{(0,0)}{\cancel{\underline{x}_0}}) +$$

$$+ \frac{1}{2} (\underline{x} - \underline{x}_0)^T \underline{\underline{H}}f(\underline{x}_0) (\underline{x} - \underline{x}_0) + R_2(\underline{x}, \underline{x}_0) =$$

$$= 0 + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \underbrace{\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{} \begin{bmatrix} x \\ y \end{bmatrix} + R_2 =$$

$$= y + \frac{1}{2} \begin{bmatrix} y & x \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + R_2 =$$

$$= y + \frac{1}{2} (yx + xy) + R_2 =$$

$$= 0 + y + xy + R_2$$

$$z = 0$$

$$z = y$$

∞

Informal proof

Let $\underline{x}_0$ be a critical point. (so $\nabla f(\underline{x}_0) = \underline{0}$)

$$f(\underline{x}) = f(\underline{x}_0) + \overset{0}{\cancel{\nabla f(\underline{x}_0)}}^T (\underline{x} - \underline{x}_0) +$$

$$+ \frac{1}{2} (\underline{x} - \underline{x}_0)^T \underline{\underline{H}}f(\underline{x}_0) (\underline{x} - \underline{x}_0) + R_2(\underline{x}, \underline{x}_0)$$

$$f(\underline{x}) - f(\underline{x}_0) = \frac{1}{2} \underbrace{(\underline{x} - \underline{x}_0)^T}_{} \underline{\underline{H}}f(\underline{x}_0) (\underline{x} - \underline{x}_0) + R_2(\underline{x}, \underline{x}_0)$$

$$\underbrace{\qquad\qquad}_{\underline{h}}$$

Since the remainder tends to 0,

$$f(\underline{x}) - f(\underline{x}_0) \approx \frac{1}{2}\underbrace{\underline{h}^T \underline{\underline{H}}_f(\underline{x}_0)\, \underline{h}}$$

If the Hessian is positive definite,

$$f(\underline{x}) - f(\underline{x}_0) \approx \frac{1}{2}\underline{h}^T \underline{\underline{H}}_f(\underline{x}_0)\, \underline{h} > 0$$

So, for any $\underline{x}$ close to $\underline{x}_0$

$$f(\underline{x}) > f(\underline{x}_0)$$

this means that $\underline{x}_0$ is a strict local minimum.

In a similar way, if $\underline{\underline{H}}_f(\underline{x}_0) \prec 0$ then $\underline{x}_0$ is a strict local maximum

If $\underline{\underline{H}}_f(\underline{x}_0)$ is indefinite, $\underline{x}_0$ is a saddle point.

$$\underline{\underline{H}}_f = \begin{bmatrix} \dfrac{4(-x^2+y^2-1)}{(x^2+y^2-1)^2} & -\dfrac{8xy}{(x^2+y^2-1)^2} \\[3em] -\dfrac{8xy}{(x^2+y^2-1)^2} & \dfrac{4(x^2-y^2-1)}{(x^2+y^2-1)^2} \end{bmatrix}$$

$A(-1, -2)$

$$\begin{bmatrix} 4\cdot 4 & 16 \end{bmatrix} \quad [\ \cdot \quad \cdot\ ]$$

$A\,(-1,\ -2)$

$$\underline{\underline{H}}R(-1,-2) = \begin{bmatrix} \frac{4\cdot 4}{16} & -\frac{16}{16} \\ -\frac{16}{16} & -\frac{16}{16} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 1 = 0$$

$$-1 - \lambda + \lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 2 = 0$$

$\lambda = \pm\sqrt{2}$ saddle point