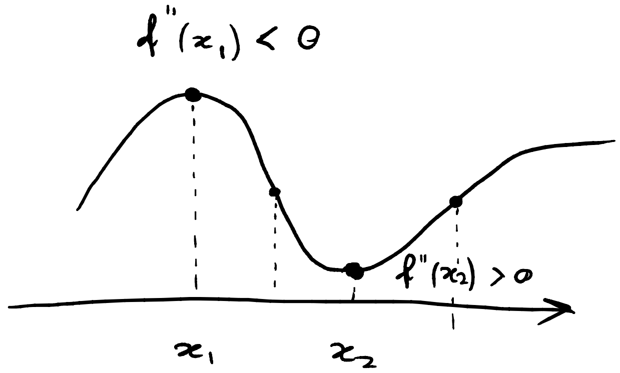
FIRST ORDER CONDITION: (necessary)

If $\underline{x}^*$ is an extreme point then $\nabla f(\underline{x}^*) = \underline{0}$

SECOND ORDER CONDITION (sufficient)

Let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^2 function and $\underline{x}^* \in A$ be an interior critical point of f.

- $\underline{\underline{H}}f(\underline{x}^*) \prec 0$ then $\underline{x}^*$ is a relative max.
- $\underline{\underline{H}}f(\underline{x}^*) \succ 0$ then $\underline{x}^*$ is a relative min.
- $\underline{\underline{H}}f(\underline{x}^*)$ is indefinite then $\underline{x}^*$ is a saddle point



$f'(x) = 0$

If $\underline{\underline{H}}f(\underline{x}^*) \succeq 0$ or $\underline{\underline{H}}f(\underline{x}^*) \preceq 0$ nothing can

If $\underline{\underline{H}}_f(\underline{x}^*) \succcurlyeq 0$ or $\underline{\underline{H}}_f(\underline{x}^*) \preccurlyeq 0$ nothing can be said about the nature of the critical point $\underline{x}^*$.

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$$f(x, y) = e^{-x^2 - y^2 - x}$$

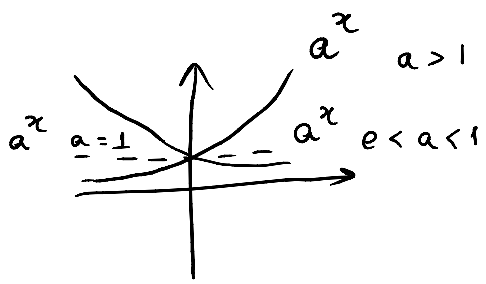
$$\text{dom } f = \mathbb{R}^2$$

$$\nabla f = \underline{0}$$

$$f'_x = e^{-x^2-y^2-x} \frac{\partial}{\partial x}(-x^2-y^2-x) =$$

$$= e^{-x^2-y^2-x}(-2x-1)$$

$$f'_y = e^{-x^2-y^2-x}(-2y)$$



$$\begin{cases} e^{-x^2-y^2-x}(-2x-1) = 0 \\ e^{-x^2-y^2-x}(-2y) = 0 \end{cases}$$

$$A \cdot B = 0 \qquad A = 0 \overset{\text{vel / or}}{\vee} B = 0$$

$$\begin{cases} -2x - 1 = 0 \\ -2y = 0 \end{cases} \qquad \begin{cases} x = -\frac{1}{2} \\ y = 0 \end{cases}$$

$$\{-2y = 0 \qquad \{ y = 0$$

$$A\left(-\frac{1}{2}, 0\right)$$

$$f''_{xx} = \frac{\partial}{\partial x}\left[e^{-x^2-y^2-x}(-2x-1)\right] =$$

$$= \frac{\partial}{\partial x}\left(e^{-x^2-y^2-x}\right)(-2x-1) +$$

$$+ e^{-x^2-y^2-x}\frac{\partial}{\partial x}(-2x-1) =$$

$$= e^{-x^2-y^2-x}(-2x-1)^2 + e^{-x^2-y^2-x}(-2) =$$

$$= e^{-x^2-y^2-x}(4x^2+4x+1-2) =$$

$$= e^{-x^2-y^2-x}(4x^2+4x-1)$$

$$\underset{\text{I}\ \ \text{II}}{f''_{xy}} = \frac{\partial^2}{\partial y\, \partial x} f = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}\left[e^{-x^2-y^2-x}(-2x-1)\right] =$$

$$= (-2x-1)\frac{\partial}{\partial y} e^{-x^2-y^2-x} =$$

$$= (-2x-1)\, e^{-x^2-y^2-x}(-2y) =$$

$$= (-2x-1)\, e^{-x^2-y^2-x} (-2y) =$$

$$= e^{-x^2-y^2-x} (2x+1)(2y)$$

$$f''_{yx} = \frac{\partial}{\partial x}\left[e^{-x^2-y^2-x} (-2y)\right] =$$

$$= -2y\, e^{-x^2-y^2-x} (-2x-1) =$$

$$= e^{-x^2-y^2-x} (2x+1)(2y)$$

$$f''_{yy} = \frac{\partial}{\partial y}\left[e^{-x^2-y^2-x} (-2y)\right] =$$

$$= \left(\frac{\partial}{\partial y} e^{-x^2-y^2-x}\right)(-2y) + e^{-x^2-y^2-x} (-2) =$$

$$= e^{-x^2-y^2-x} (-2y)(-2y) + e^{-x^2-y^2-x} (-2) =$$

$$= e^{-x^2-y^2-x} (4y^2-2)$$

$A\left(-\frac{1}{2}, 0\right)$

$$e^{-x^2-y^2-x} = e^{-\left(-\frac{1}{2}\right)^2 - 0^2 - \left(-\frac{1}{2}\right)} =$$

$$= e^{-\frac{1}{4}+\frac{1}{2}} = e^{-\frac{1}{4}+\frac{2}{4}} = e^{\frac{1}{4}} = \sqrt[4]{e}$$

$$f''_{xx} = e^{-x^2-y^2-x}\left(4x^2+4x-1\right)$$

$$f''_{xx}\left(-\frac{1}{2}, 0\right) = \sqrt[4]{e}\left(4\left(-\frac{1}{2}\right)^2+4\left(-\frac{1}{2}\right)-1\right) =$$

$$= \sqrt[4]{e}\left(\cancel{4}\cdot\frac{1}{\cancel{4}} - 2 \cancel{-1}\right) = -2\sqrt[4]{e}$$

$$f''_{xy} = f''_{yx} = e^{-x^2-y^2-x}\left(2x+1\right)\left(2y\right)$$

$$f''_{xy}(A) = f''_{yx}(A) = \sqrt[4]{e}\left(2\left(-\frac{1}{2}\right)+1\right)\left(2\cdot 0\right) = 0$$

$$f''_{yy} = e^{-x^2-y^2-x}\left(4y^2-2\right)$$

$$f''_{yy}(A) = \sqrt[4]{e}\,(-2) = -2\sqrt[4]{e}$$

$$\underline{\underline{H}}f(A) = \begin{bmatrix} -2\sqrt[4]{e} & 0 \\ 0 & -2\sqrt[4]{e} \end{bmatrix}$$

$\det\left(\underline{\underline{H}} - \lambda \underline{\underline{I}}\right) = 0$ not necessary here

There are two eigenvalues equal to $-2\sqrt[4]{e} < 0$

So $\underline{\underline{H}}f(A) \prec 0$

So $\underline{\underline{H}}f(A) \prec 0$

So A is a relative maximum

$$f(x,y) = x^4 + y^4 - 4xy$$

$$\operatorname{dom} f = \mathbb{R}^2$$

$$\nabla f = \underline{0}$$

$$f'_x = 4x^3 - 4y$$

$$f'_y = 4y^3 - 4x$$

$$\begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases} \qquad \begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases} \qquad \begin{cases} y = x^3 \\ y^3 - x = 0 \end{cases}$$

$$\begin{cases} y = x^3 \\ (x^3)^3 - x = 0 \end{cases} \qquad \begin{cases} y = x^3 \\ x^9 - x = 0 \end{cases}$$

$$x^9 - x = x(x^8 - 1) =$$

$$= x\left((x^4)^2 - 1^2\right) =$$

$$= x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1) =$$

$$= x(x-1)(x+1)\underline{(x^2+1)}\,\underline{(x^4+1)} = 0$$

$$\begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} y = 1 \\ x = 1 \end{cases} \vee \begin{cases} y = -1 \\ x = -1 \end{cases}$$

$O(0, 0) \qquad A(-1, -1) \qquad B(1, 1)$

$f'_x = 4x^3 - 4y \qquad f'_y = 4y^3 - 4x$

$f''_{xx} = 12x^2 \qquad f''_{xy} = -4$

$f''_{yx} = -4 \qquad f''_{yy} = 12y^2$

$$\underline{\underline{H}}f = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

$$\underline{\underline{H}}f(O) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

$$\det\left(\underline{\underline{H}}_f(O) - \lambda \underline{\underline{I}}\right) = 0$$

$$\begin{vmatrix} -\lambda & -4 \\ -4 & -\lambda \end{vmatrix} = 0$$

$-$

$$(-\lambda)(-\lambda) - (-4)(-4) = 0$$

$\lambda^2 - 16 = 0 \qquad \lambda^2 = 16 \qquad \lambda = \pm 4$

indefinite

$O(0, 0)$ saddle point

$A(-1, -1) \qquad B(1, 1)$

$$\underline{\underline{H}}_f = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

$$\underline{\underline{H}}_f(A) = \underline{\underline{H}}_f(B) = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}$$

$$\begin{vmatrix} 12-\lambda & -4 \\ -4 & 12-\lambda \end{vmatrix} = 0$$

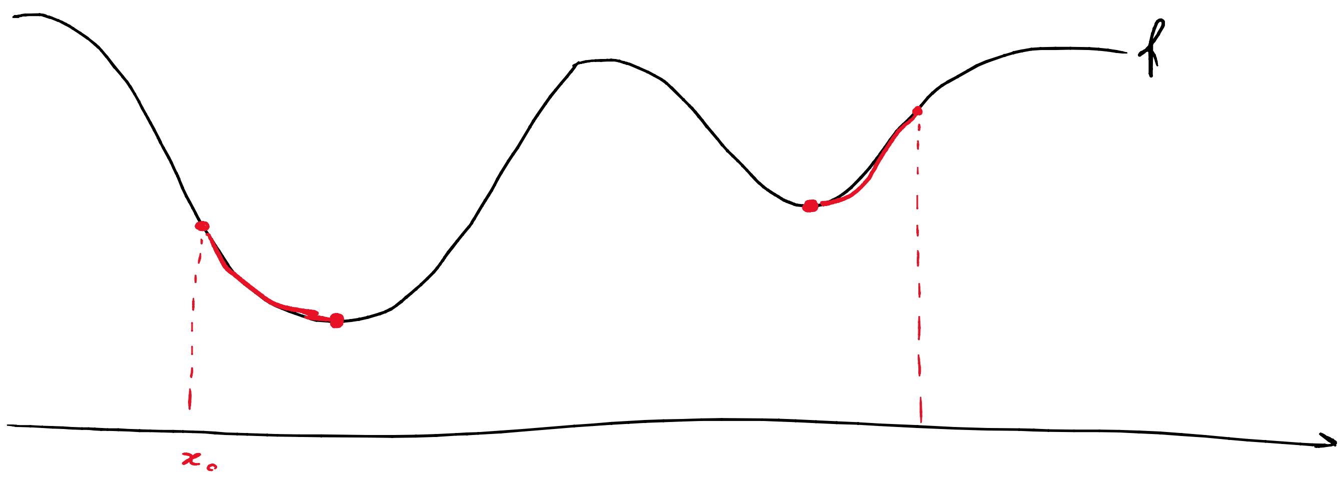
$$(12-\lambda)^2 - 16 = 0$$

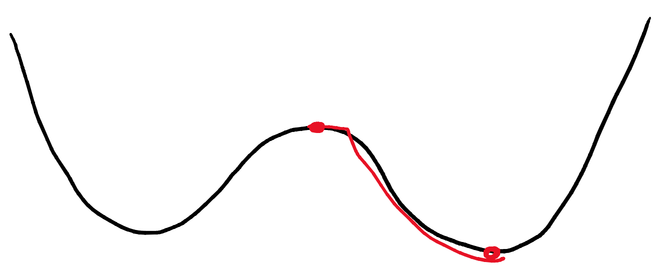
$$(\lambda - 12)^2 = 16$$

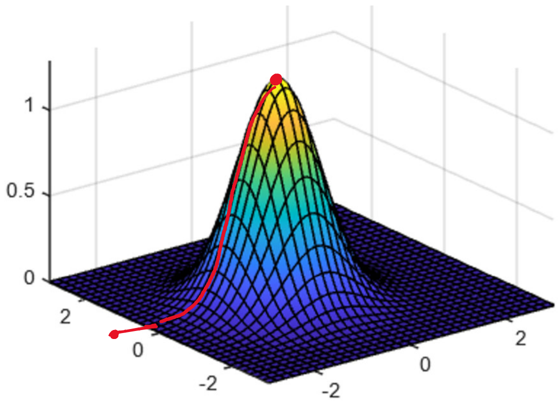
$$\lambda - 12 = \pm 4$$

$$\lambda = 12 \pm 4 \left\langle \begin{array}{l} 16 \\ 8 \end{array} \right.$$

positive definite so
A and B are local minima






1
0.5
0
2
0
-2
-2
0
2