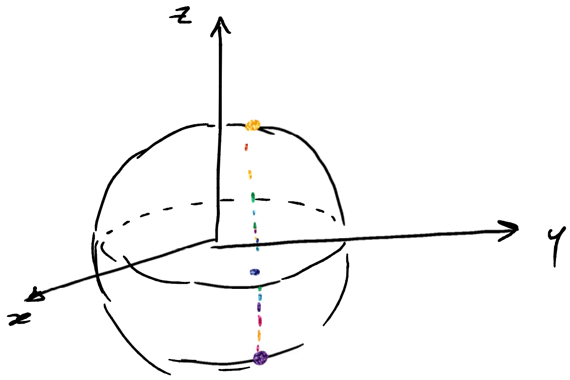
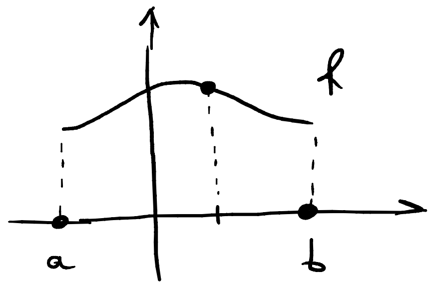
$$x^2 + y^2 + z^2 = 9$$

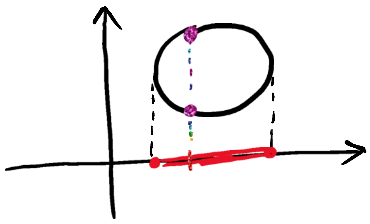
sphere with center $(0, 0, 0)$ and radius 3





$\operatorname{dom} f = [a, b]$

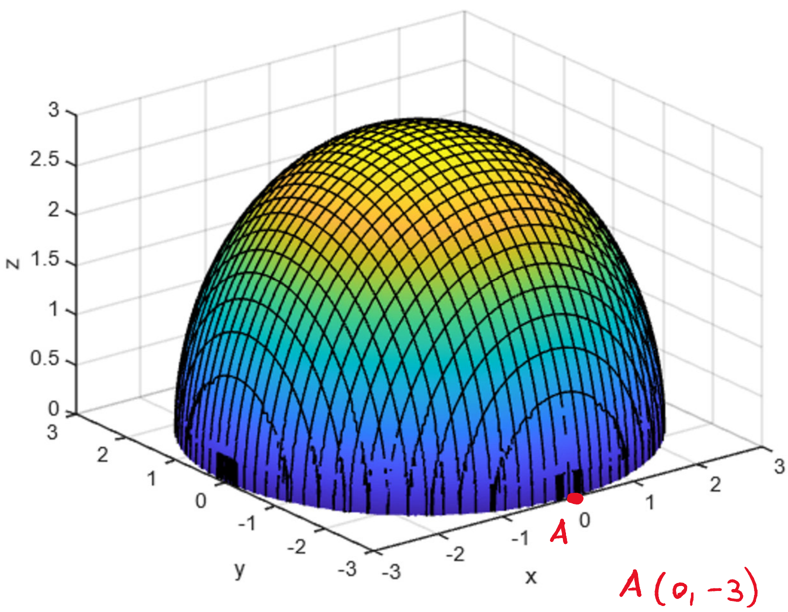
This is not a function but



$$x^2 + y^2 + z^2 = 9$$

$$z^2 = 9 - x^2 - y^2$$

$$z = \pm\sqrt{9 - x^2 - y^2}$$



$$z = \sqrt{9 - x^2 - y^2}$$

$$f'_x = \frac{1}{2\sqrt{9 - x^2 - y^2}} \, (-2x) =$$

$$= - \frac{x}{\sqrt{9 - x^2 - y^2}}$$

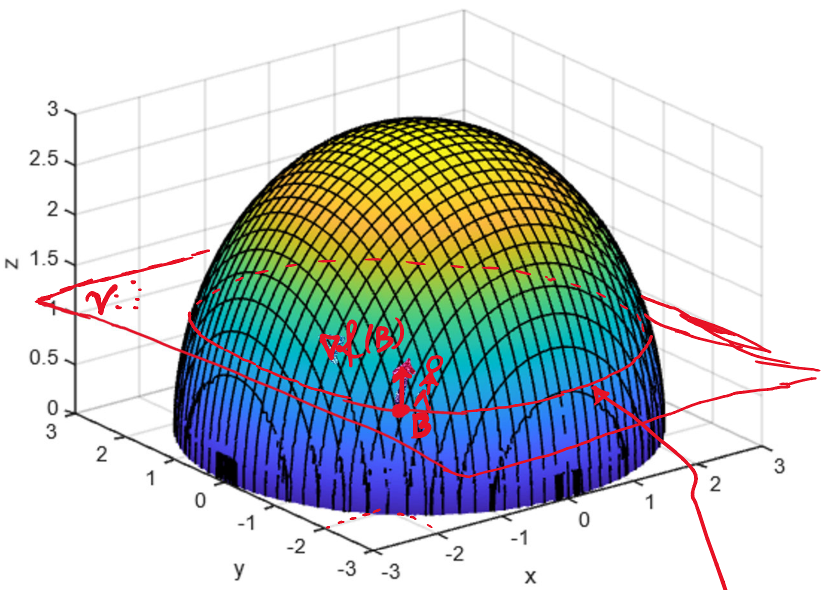
$$f'_y = - \frac{y}{\sqrt{9 - x^2 - y^2}}$$

$$f'_x(A) = f'_x(0, -3) = 0$$

$f'_y(A)$ is not defined.

$$D\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$D\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \, f'(x)$$



1) The gradient gives you the direction of maximum growth

plane of equation $z = 1$

level set for $z = 1$

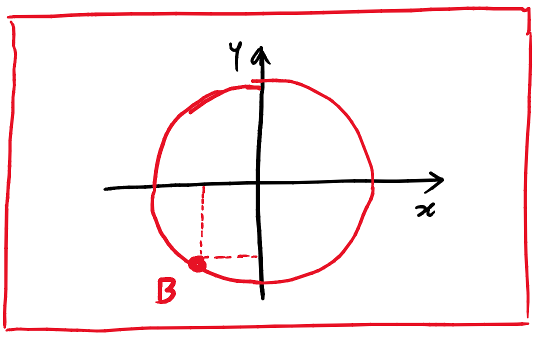
$$\begin{cases} z = \sqrt{9 - x^2 - y^2} \\ z = 1 \end{cases}$$

$$1 = \sqrt{9 - x^2 - y^2}$$

$$1 = \sqrt{9 - x^2 - y^2}$$

$$1 = 9 - x^2 - y^2$$

$$x^2 + y^2 = 8 \quad \text{this is the level set}$$



$z = 1$

plane γ

$B(-2, -2)$

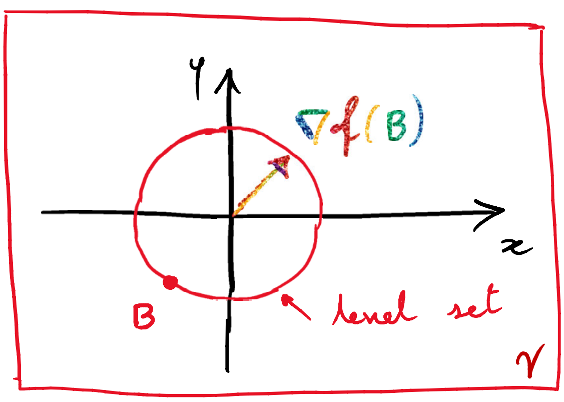
$$f'_x = -\frac{x}{\sqrt{9 - x^2 - y^2}}$$

$$f'_y = -\frac{y}{\sqrt{9 - x^2 - y^2}}$$

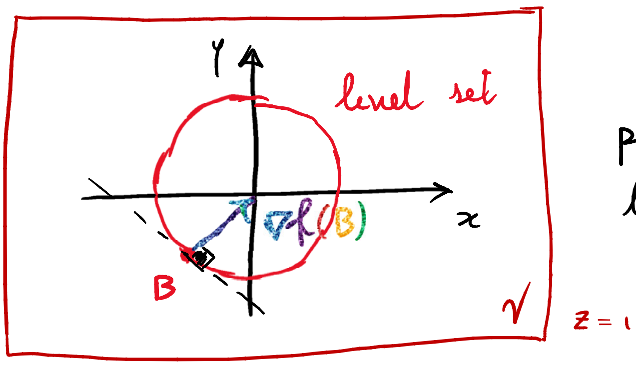
$$f'_x(B) = f'_x(-2, -2) = -\frac{-2}{1} = 2$$

$$f'_y(B) = 2$$

$$\nabla f(B) = (2, 2)$$



$z = 1$

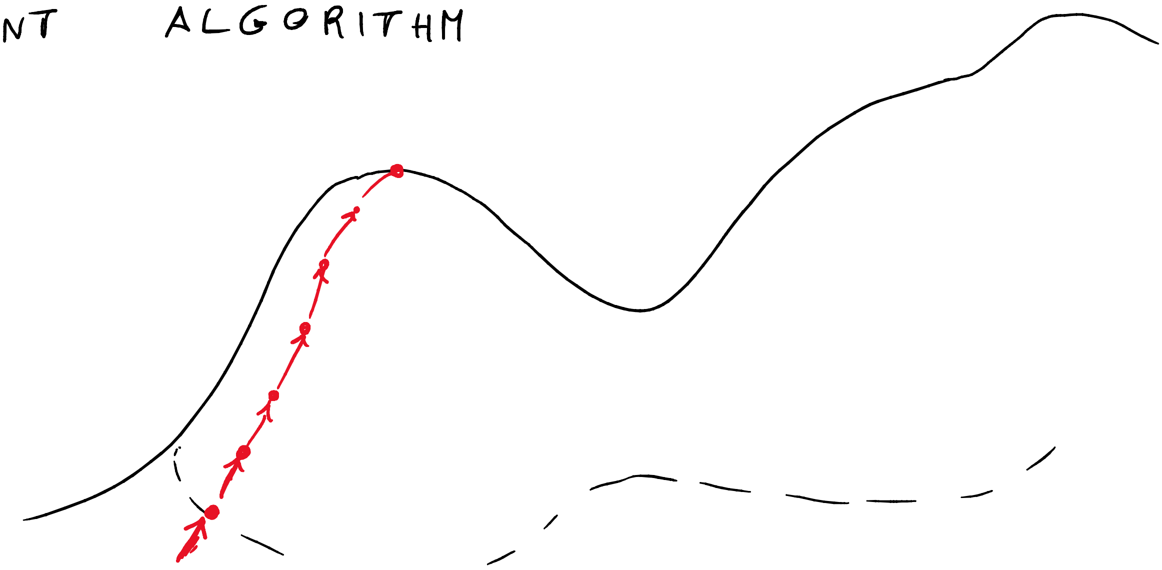


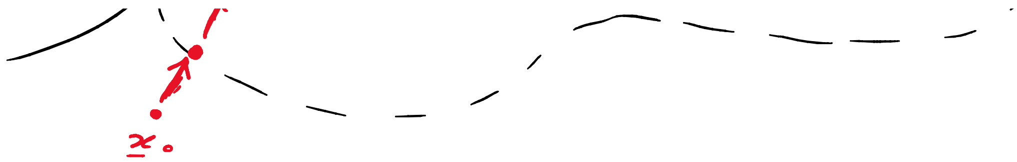
2) the gradient is perpendicular to the level set.

THEOREM 1: let $f: A \subseteq \mathbb{R}^m \to \mathbb{R}$ be a C^1 function. At any regular point $\underline{x}$ inside A at which $\nabla f(\underline{x}) \neq \underline{0}$, the gradient $\nabla f(\underline{x})$ points at $\underline{x}$ into the direction in which f increases most rapidly.

THEOREM 2: let $f: A \subseteq \mathbb{R}^m \to \mathbb{R}$ be a C^1 function and let $\underline{x}_0$ be an interior point of A such that $\nabla f(\underline{x}_0) \neq \underline{0}$. Then the gradient vector $\nabla f(\underline{x}_0)$ is perpendicular to the level set $f(x_1, x_2, \dots, x_m) = z$ at $\underline{x}_0$

GRADIENT ALGORITHM





- choose an initial point $\underline{x}_0$
- compute the gradient at $\underline{x}_0$
- you move "a little bit" far from $\underline{x}_0$ towards the direction of the gradient
- repeat.

I reach the maximum if $\nabla f = \underline{0}$

$$f(\underline{x}) = 1 \cdot 10^{-12}$$

$$\| \nabla f \| < \varepsilon \leftarrow \text{threshold}$$

↑ it is the length of the vector.

$$f : A \subseteq \mathbb{R}^n \to \mathbb{R}, \quad f \in C^2(A)$$

$$f(\underline{x}) = f(\underline{x}_0) + \nabla f(\underline{x}_0)^T (\underline{x} - \underline{x}_0) +$$

$$+ \frac{1}{2} (\underline{x} - \underline{x}_0)^T \underline{\underline{H}}_f(\underline{x}_0)(\underline{x} - \underline{x}_0) + R_2(\underline{x}, \underline{x}_0)$$

$$\lim_{\underline{x} \to \underline{x}_0} \frac{R_2(\underline{x}, \underline{x}_0)}{\| \underline{x} - \underline{x}_0 \|^2}$$

$$Q(\underline{x}) = 5x_1^2 + 4x_2^2 + 3x_3^2 + 2x_1 x_2 +$$

$$+ 6x_1x_3 + 4x_2x_3 =$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 5 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & 3 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\underline{x}^T \underline{\underline{A}} \underline{x}$$

$$\frac{\partial Q}{\partial x_1} = 10x_1 + 2x_2 + 6x_3$$

$$\frac{\partial Q}{\partial x_2} = 8x_2 + 2x_1 + 4x_3$$

$$\frac{\partial Q}{\partial x_3} = 6x_3 + 6x_1 + 4x_2$$

$$\nabla Q = \underline{0} \qquad \begin{cases} 10x_1 + 2x_2 + 6x_3 = 0 \\ 8x_2 + 2x_1 + 4x_3 = 0 \\ 6x_3 + 6x_1 + 4x_2 = 0 \end{cases}$$

$$(x_1, x_2, x_3) = (0, 0, 0)$$

$$\frac{\partial Q}{\partial x_1} = 10x_1 + 2x_2 + 6x_3$$

$$\frac{\partial Q}{\partial x_2} = 8x_2 + 2x_1 + 4x_3$$

$$\frac{\partial Q}{\partial x_3} = 6x_3 + 6x_1 + 4x_2$$

$$Q''_{x_1x_1} = 10 \qquad Q''_{x_1x_2} = 2 \qquad \underline{\underline{H}}_Q = \begin{bmatrix} 10 & 2 & 6 \\ 8 & 2 & 4 \\ 6 & 6 & 4 \end{bmatrix}$$

$$\underline{x}_0 = \underline{0} \qquad \nabla f(\underline{x}_0) = \underline{0} \qquad f(\underline{0}) = 0$$

$$\underline{x}_0 = \underline{0} \qquad \nabla f(\underline{x}_0) = \underline{0} \qquad f(\underline{0}) = 0$$

$$f(\underline{x}) = \cancel{f(\underline{x}_0)} + \cancel{\nabla f(\underline{x}_0)^T(\underline{x} - \underline{x}_0)} +$$

$$+ \frac{1}{2}(\underline{x} - \cancel{\underline{x}_0})^T \underline{\underline{H}}_f(\underline{x}_0)(\underline{x} - \cancel{\underline{x}_0}) + R_2(\underline{x}, \underline{x}_0) =$$

$$= \underline{x}^T \frac{1}{2} \underline{\underline{H}}_Q \underline{x} = \underline{x}^T \begin{bmatrix} 5 & 1 & 3 \\ 4 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix} \underline{x} + \underbrace{R_2(\underline{x}, \underline{0})}_{= 0}$$

We get the same result because the quadratic form is a particular polynomial of degree 2 that we are approximating with another polynomial of degree 2 (the Taylor polynomial). Therefore, we get the same result.