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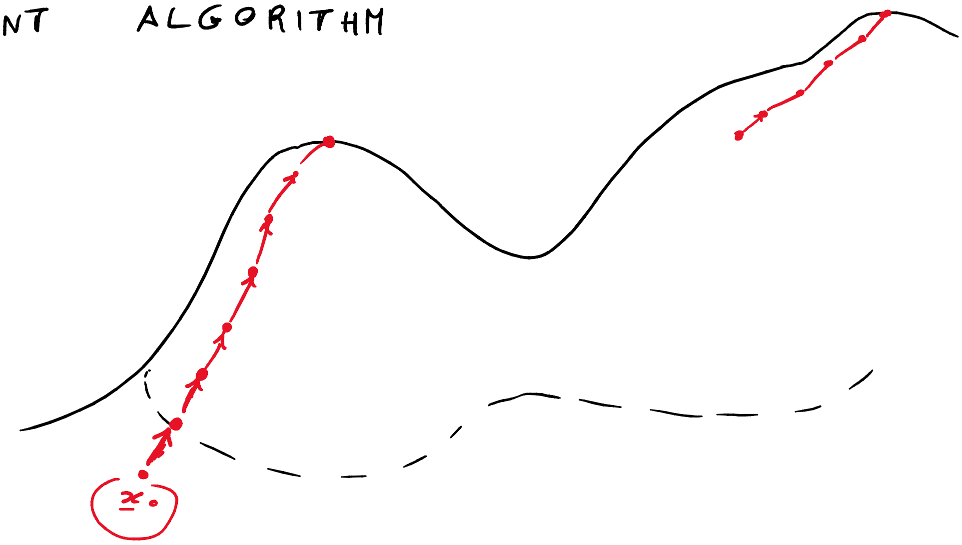
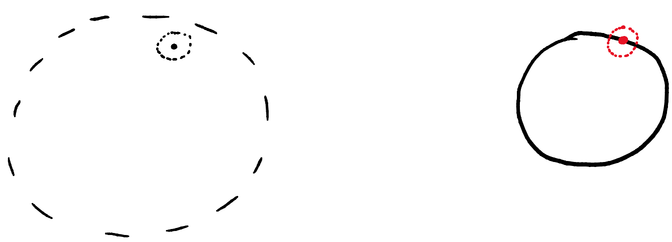
THEOREM 1: Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. At any regular point $\underline{x}$ inside A at which $\nabla f(\underline{x}) \neq \underline{0}$, the gradient $\nabla f(\underline{x})$ points at $\underline{x}$ into the direction in which f increases most rapidly.

THEOREM 2: Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and let $\underline{x}_0$ be an interior point of A such that $\nabla f(\underline{x}_0) \neq \underline{0}$. Then the gradient vector $\nabla f(\underline{x}_0)$ is perpendicular to the level set $f(x_1, x_2, \dots, x_n) = z$ at $\underline{x}_0$

GRADIENT ALGORITHM



OPEN SETS



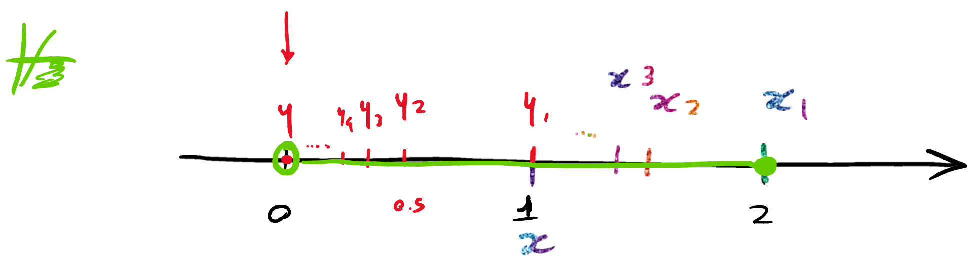
CLOSED SETS

A set $S \subseteq \mathbb{R}^m$ is closed if, whenever $\{\underline{x}_m\}_{m=1}^{+\infty}$ is a convergent sequence completely contained in S, its limit is also contained in S.

$$x_m = 1 + \frac{1}{m} \qquad\qquad y_m = \frac{1}{m}$$

m	x_m	y_m
1	2	1
2	1.5	0.5
3	1.333	0.333 ....
4	1.25	0.25
5	1.2	0.2
$\vdots$		

the limit of $\{y_m\}_{m=1}^{+\infty}$



$$A = (0, 2]$$

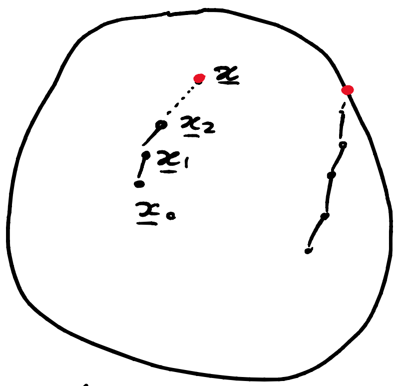
$\{x_m\}_{m=1}^{+\infty}$ is completely contained in A and also its limit (that is 1)

$\{y_m\}^{+\infty}$ is completely contained in A but not its

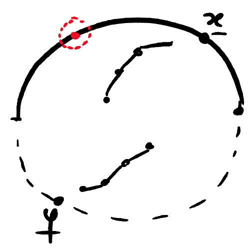
$\{y_n\}_{n=1}^{+\infty}$ is completely contained in A but not its limit (that is 0) so A is not a closed set.

But if we consider $B = [0, 2]$ then it is closed.

[ ] ( )



closed



not closed

and not open

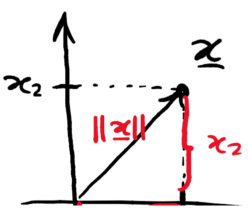
NORM

$\underline{x}, \underline{y} \in \mathbb{R}^n$

$$d(\underline{x}, \underline{y}) = \| \underline{x} - \underline{y} \| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2}$$

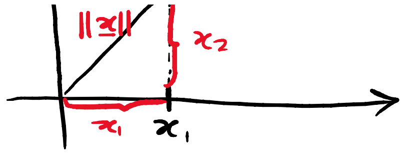
$$\| \underline{x} \| = \| \underline{x} - \underline{0} \| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$\mathbb{R}^2$



$\underline{x} = (x_1, x_2)$

$$\| \underline{x} \| = \sqrt{x_1^2 + x_2^2}$$

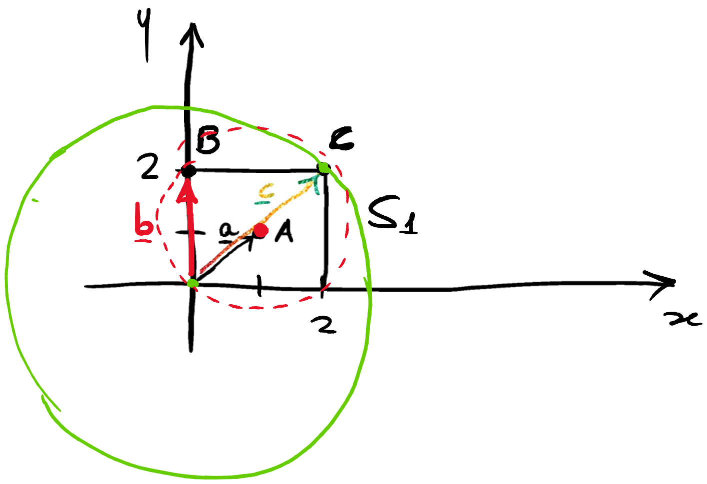


$$\|\underline{x}\| = \sqrt{x_1^2 + x_2^2}$$

So the norm of a vector is equal to its length

BOUNDED SETS

A set $S \subseteq \mathbb{R}^m$ is bounded if there exists a number B such that $\|\underline{x}\| \leq B \quad \forall\, \underline{x} \in S$, that is, if S is contained in some ball.



$\underline{a} = (1, 1)$

$\|\underline{a}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\underline{b} = (0, 2)$

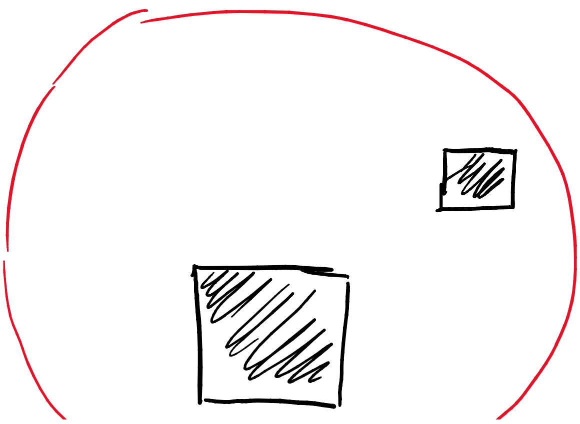
$\|\underline{b}\| = 2$

$\underline{c} = (2, 2)$

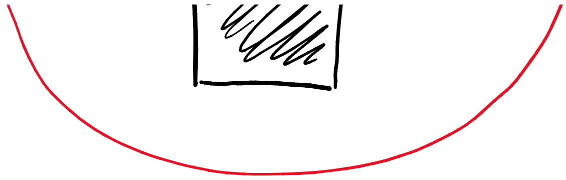
$\|\underline{c}\| = \sqrt{8} = \underbrace{2\sqrt{2}}_{B}$

For any point $\underline{x} \in S_1$, $\|\underline{x}\| \leq 2\sqrt{2}$

so S_1 is bounded

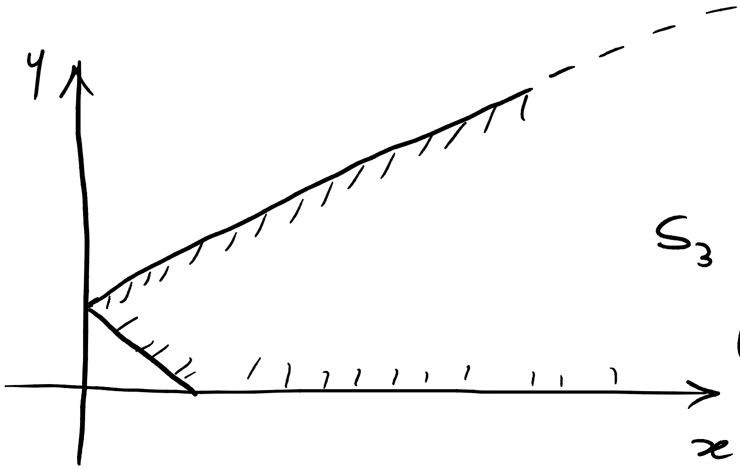


S_2



S_2

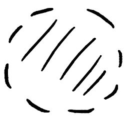
bounded



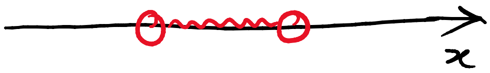
S_3 is not bounded
(it is unbounded)

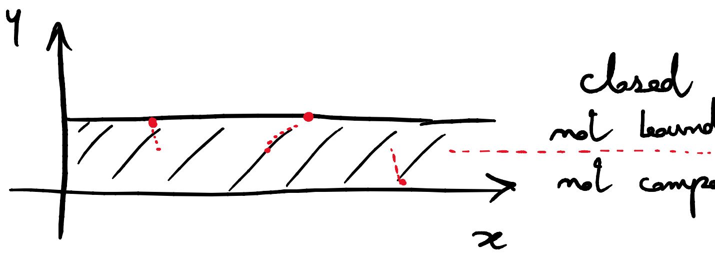
COMPACT SETS

A set $S \subseteq \mathbb{R}^n$ is compact if and only if it is both closed and bounded.

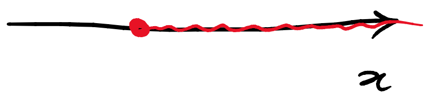


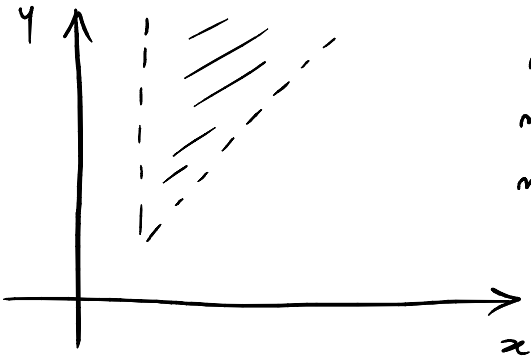
not closed
bounded
not compact



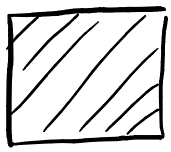


closed
not bounded
not compact





not closed
not bounded
not compact



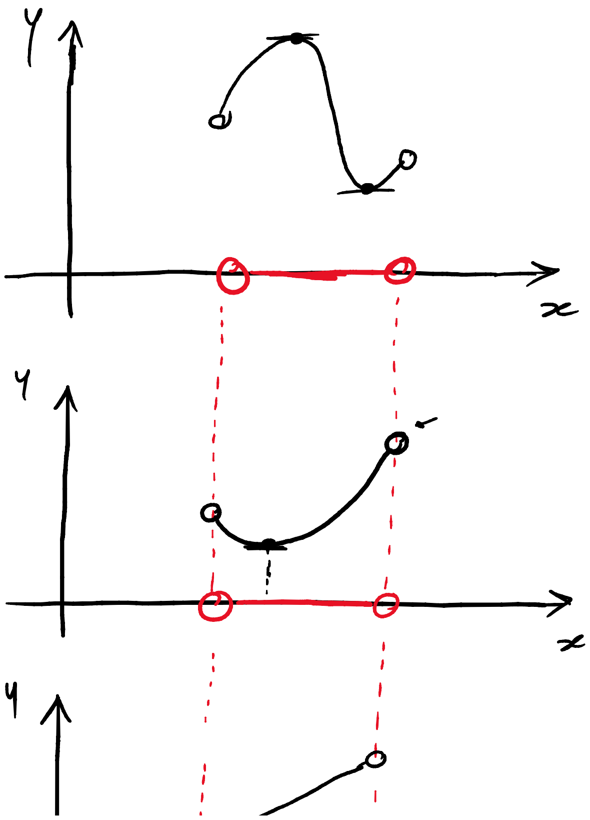
closed
bounded
compact

WEIERSTRASS (WEIERSTRAß)'s THEOREM

Let $f : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function whose domain A is a compact set. Then, there exist points $\underline{x}_m$ and $\underline{x}_M$ in A such that

$$f(\underline{x}_m) \leqslant f(\underline{x}) \leqslant f(\underline{x}_M) \quad \forall \, \underline{x} \in A.$$

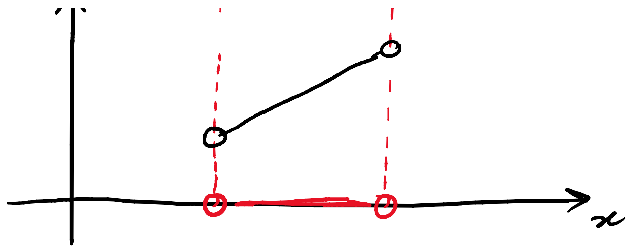
That is, $\underline{x}_m \in A$ is a global minimum of f in A and $\underline{x}_M \in A$ is a global maximum of f in A.



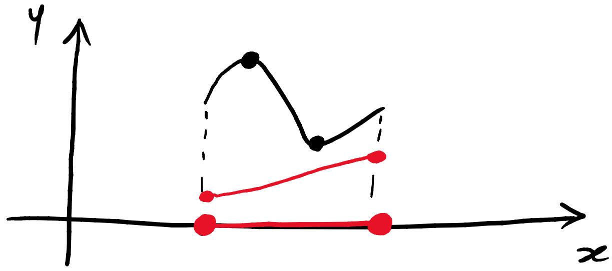
f has a minimum and a maximum
A is not compact

f has a minimum but not a maximum
A is not compact

f does not have



f does not have minimum nor maximum



A is compact, f is continuous. There is a minimum and a maximum.

CONSTRAINED OPTIMIZATION PROBLEMS

$$\max \ f(x_1, x_2, \dots, x_n)$$

$$\text{s.t.:} \quad g_1(x_1, x_2, \dots, x_n) \leq b_1$$

$$\vdots$$

$$g_k(x_1, x_2, \dots, x_n) \leq b_k$$

K inequality constraints

$$h_1(x_1, x_2, \dots, x_n) = c_1$$

$$\vdots$$

$$h_m(x_1, x_2, \dots, x_n) = c_m$$

m equality constraints

utility maximization problem

$$U(x_1, x_2, \dots, x_n)$$

x_i : amount of commodity i $\quad i \in \{1, 2, \dots, n\}$

x_i : amount of commodity i $\quad i \in \{1, 2, \dots, n\}$
variables

p_i : unit price of commodity i
data , constant

I : individual's budget.

pasta	x_1	p_1
pâté	x_2	p_2
raclette	x_3	p_3
hummus	x_4	$\vdots$
beans	x_5	
empanadas	$\vdots$	
ingjera		
soup ←		

$$\max \quad U(x_1, x_2, \dots, x_n)$$

$$\text{s.t.:} \quad p_1 x_1 + p_2 x_2 + \dots + p_n x_n \leq I$$ budget constraint

$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$\vdots$$
$$x_n \geq 0$$
non-negativity constraints.

$$\min \quad g(\underline{x}) \qquad\qquad \max \quad -g(\underline{x})$$

$$2x_1 + x_2 \geq 3$$
$$\downarrow$$

$$-2x_1 - x_2 \leq -3$$

Profit maximization of a firm.

A firm uses n inputs to produce a final product. The amount y of the final product is given by $y = f(x_1, x_2, \dots, x_n)$

P: unit price of the final product

w_i: unit cost of input i

$$\max \quad Py - \sum_{i=1}^{n} w_i x_i$$

$$\text{s.t.:} \quad Py - \sum_{i=1}^{n} w_i x_i \geq 0$$

$$y - f(x_1, x_2, \dots, x_n) = 0$$

$$q_1(x_1, x_2, \dots, x_n) \leq b_1$$

$$\vdots$$

$$q_K(x_1, x_2, \dots, x_n) \leq b_K$$

constraints on the availability of the inputs

$$x_1 \geq 0$$

$$\vdots$$

$$x_n \geq 0$$

$$\max \quad z = xy$$

$$\text{s.t.:} \quad x^2 + y^2 = 1$$

level sets: I fix a constant k, I vary it and I get a family of curves.

$$k = xy$$

$$y = \frac{k}{x}$$