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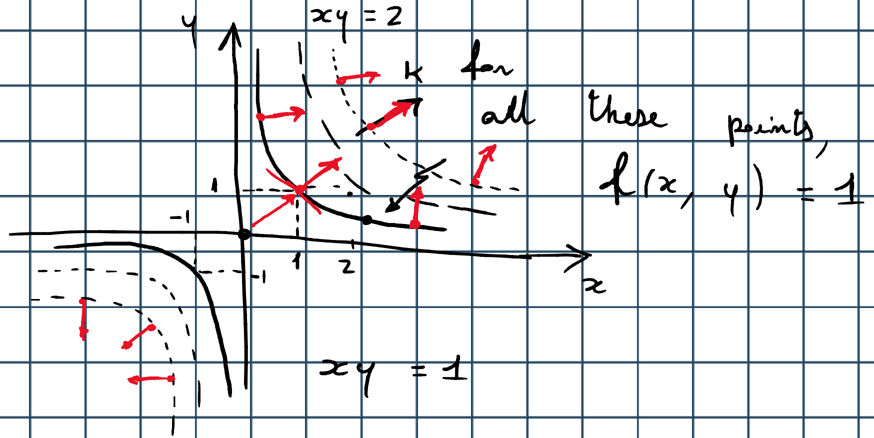
THEOREM 2: let $f : A \subseteq \mathbb{R}^m \to \mathbb{R}$ be a $C^1$ function and let $\underline{x}_0$ be an interior point of $A$ such that $\nabla f(\underline{x}_0) \neq \underline{0}$. Then the gradient vector $\nabla f(\underline{x}_0)$ is perpendicular to the level set $f(x_1, x_2, \dots, x_m) = z$ at $\underline{x}_0$

$$\max \quad z = xy$$
$$\text{s.t.}: \quad x^2 + y^2 = 1$$

$$f(x, y) = xy$$
$$k = xy$$
$$y = \frac{k}{x}$$




Since we are in a maximization problem, we want k to be as big as possible

$$f(x, y) = xy$$
$$\nabla f = (y, x)$$

$$\nabla f = (y, x)$$

$$\nabla f(1, 1) = (1, 1)$$

$$h(x, y) = x^2 + y^2$$

The level set of this function is:

$$K = x^2 + y^2$$

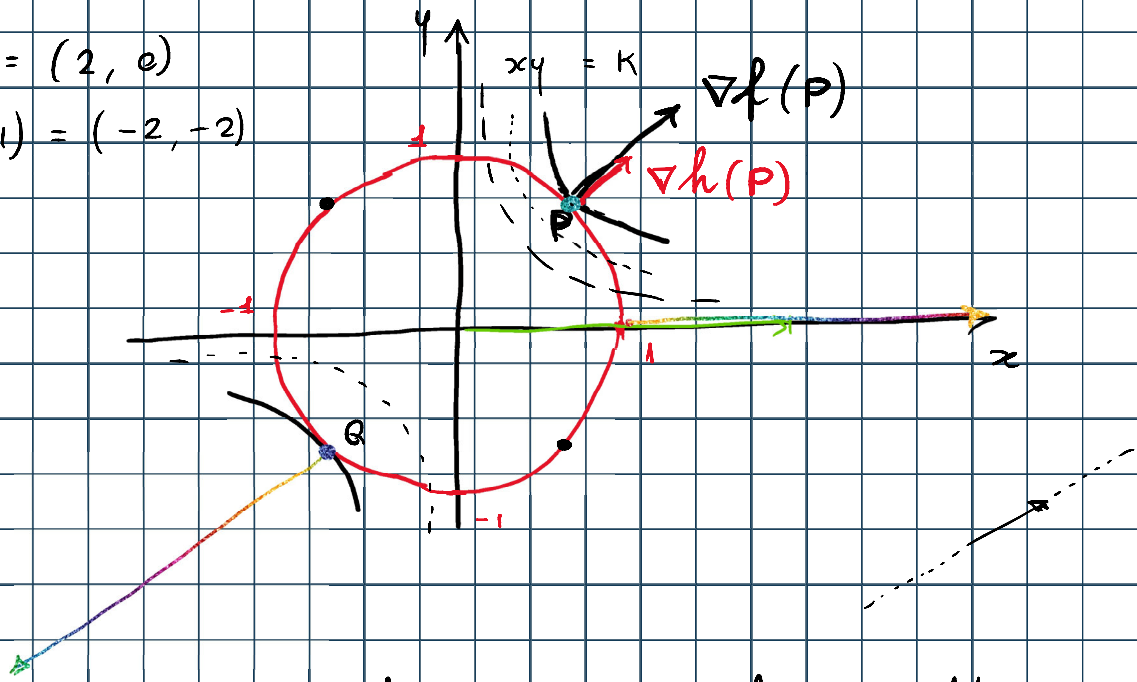
Here $K = 1$ because the constraint is $x^2 + y^2 = 1$

This means that our constraint is a particular level curve of the function $h(x, y) = x^2 + y^2$ with $K = 1$.

$$\nabla h = (2x, 2y)$$

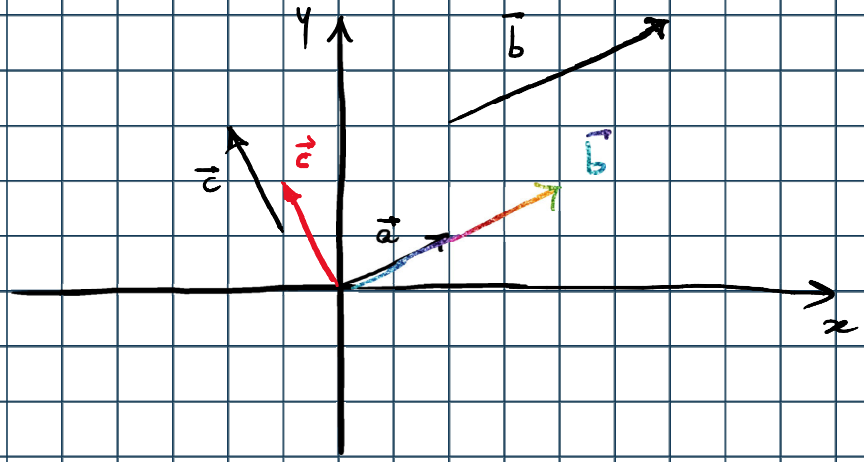
$\nabla h(1, 0) = (2, 0)$

$\nabla h(-1, -1) = (-2, -2)$




If we take the highest value of K, the corresponding level curve of f will be tangent to the constraint.

At point P (and also Q) $\nabla f$ is parallel to $\nabla h$.




$\vec{a} = (2, 1)$ $\qquad$ $\vec{b} = (4, 2)$ $\qquad$ $\vec{c} = (-1, 2)$

$\vec{b} = (4, 2) = 2(2, 1) = 2\vec{a}$

$\vec{a} \parallel \vec{b} \Leftrightarrow \exists \ \alpha \neq 0 : \quad \vec{b} = \alpha \vec{a}$

So, since $\nabla f(P) \parallel \nabla h(P)$, then there exists $\mu$ such that $\boxed{\nabla f(P) = \mu \nabla h(P)}$

$\mu$ : Lagrange multiplier

$$\frac{\partial f}{\partial x}(P) = \mu \frac{\partial h}{\partial x}(P)$$

$$\frac{\partial f}{\partial y}(P) = \mu \frac{\partial h}{\partial y}(P)$$

Lagrangian function $\qquad$ Lagrange

$$\mathcal{L}(x, y, \mu) = f(x, y) - \mu\left(h(x, y) - c\right)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \mu \frac{\partial h}{\partial x} = 0 \longrightarrow \frac{\partial f}{\partial x} = \mu \frac{\partial h}{\partial x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} - \mu \frac{\partial h}{\partial y} = 0 \longrightarrow \frac{\partial f}{\partial y} = \mu \frac{\partial h}{\partial y}$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = -h(x,y) + c = 0 \longrightarrow h(x,y) = c$$

Theorem: let $f$ and $h$ be $C^1$ functions of two variables. Suppose that $\underline{x}^* = (x^*, y^*)$ is a solution of the problem

$$\max \ f(x, y)$$

$$\text{s.t.: } h(x, y) = c$$

Suppose further that $(x^*, y^*)$ is not a critical point of $h$ (that is $\nabla h(x^*, y^*) \neq (0, 0)$). Then, there is a real number $\mu^*$ such that $(x^*, y^*, \mu^*)$ is a critical point of the Lagrangian function

$$\mathcal{L}(x, y, \mu) = f(x, y) - \mu \left( h(x, y) - c \right)$$

In other, at $(x^*, y^*, \mu^*)$

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = 0, \quad \frac{\partial \mathcal{L}}{\partial \mu} = 0.$$

$$\max \ z = xy$$

$$\text{s.t. } x^2 + y^2 = 1$$

$$f(x,y) = xy \qquad \nabla f = (y, x)$$

$$h(x,y) = x^2+y^2 \qquad \nabla h = (2x, 2y)$$

CONSTRAINT QUALIFICATION: $\nabla h \neq (0,0)$

$\nabla h = (0,0)$ when $(x,y) = (0,0)$

$$x^2+y^2 = 1$$

$(0,0)$ does not belong to the constraint

$$\mathcal{L}(x,y,\mu) = f(x,y) - \mu\left(h(x,y) - c\right) =$$

$$= xy - \mu\left(x^2+y^2-1\right)$$

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2x\mu = 0$$

$$\begin{cases} y - 2x\mu = 0 \\ x - 2y\mu = 0 \\ x^2+y^2 = 1 \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - 2y\mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = -\left(x^2+y^2-1\right) = 0$$

$$\begin{cases} y = 2x\mu \\ x = 2y\mu \\ x^2+y^2 = 1 \end{cases}$$

$$y = 2x\mu = 2(2y\mu)\mu$$

$$y = 4\mu^2 y$$

$$y(1-4\mu^2) = 0$$

$$y(1-4\mu^2) = 0$$

$$y(1-2\mu)(1+2\mu) = 0$$

$$\begin{cases} y(1-2\mu)(1+2\mu) = 0 \\ x = 2y\mu \\ x^2 + y^2 = 1 \end{cases}$$

$$\cancel{\begin{cases} y = 0 \\ x = 0 \\ 0 = 1 \end{cases}} \text{ imp.} \quad \vee \quad \begin{cases} \mu = \frac{1}{2} \\ x = y \\ 2y^2 = 1 \end{cases} \quad \begin{cases} \mu = \frac{1}{2} \\ x = \frac{\sqrt{2}}{2} \\ y = \frac{\sqrt{2}}{2} \end{cases} \vee \begin{cases} \mu = \frac{1}{2} \\ x = -\frac{\sqrt{2}}{2} \\ y = -\frac{\sqrt{2}}{2} \end{cases} \vee$$

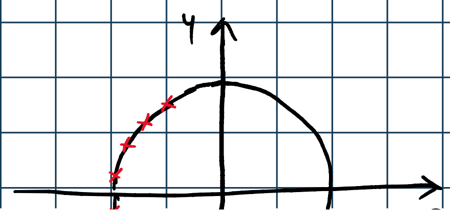
$$\begin{cases} \mu = -\frac{1}{2} \\ x = -y \\ 2y^2 = 1 \end{cases} \quad \begin{cases} \mu = -\frac{1}{2} \\ x = -\frac{\sqrt{2}}{2} \\ y = \frac{\sqrt{2}}{2} \end{cases} \vee \begin{cases} \mu = -\frac{1}{2} \\ x = \frac{\sqrt{2}}{2} \\ y = -\frac{\sqrt{2}}{2} \end{cases}$$

$$A\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad B\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \quad C\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad D\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

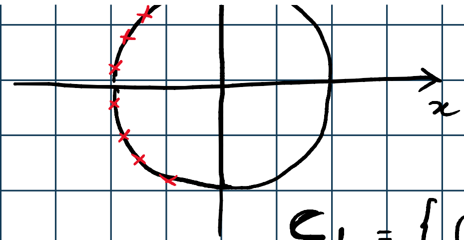
The theorem above gives us NECESSARY conditions, that is, if $(x^*, y^*, \mu^*)$ is an extreme point then $\nabla \mathcal{L}(x^*, y^*, \mu^*)$ so this means that if we are looking for the maximum (or the minimum), we consider the critical points of the Lagrangian function but this does not necessarily means that those points are maxima (or minima).

So we need additional considerations to say whether a critical point of the Lagrangian is a maximum or a minimum.

$$x^2 + y^2 = 1$$



$x + y = 1$




$C_h = \{(x,y) \in \mathbb{R} : x^2 + y^2 = 1\}$

$C_h$ is bounded and closed so it is compact

$f(x,y) = xy$ is continuous.

So, by Weierstrass's theorem, there is global minimum and maximum of $f$ over $C_h$.

That is, at least one of the points A, B, C, D is a maximum and one is a minimum.

$A\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $B\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $C\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $D\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$f(A) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2} = f(B)$

$f(C) = -\frac{1}{2} = f(D)$

A, B are maximum points
C and D are minimum points.

max $z = xy$

s.t.: $x + 4y = 16$

$h(x,y) = x + 4y$

$\nabla h = (1, 4)$ is always $\neq (0,0)$

$$\mathcal{L} = xy - \mu(x + 4y - 16)$$

$$\frac{\partial \mathcal{L}}{\partial x} = y - \mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - 4\mu = 0$$

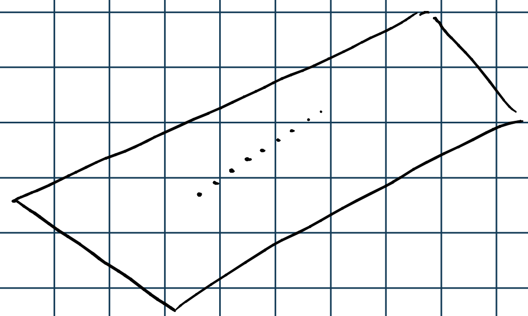
...

$$\begin{cases} y - \mu = 0 \\ x - 4\mu = 0 \\ x + 4y = 16 \end{cases} \qquad \begin{cases} y = \mu \\ x = 4\mu \\ 4\mu + 4\mu = 16 \end{cases} \qquad \begin{cases} y = \mu \\ x = 4\mu \\ 8\mu = 16 \end{cases}$$

$$\begin{cases} y = 2 \\ x = 8 \\ \mu = 2 \end{cases} \qquad A(8, 2)$$

$x + 4y = 16$

closed but
not bounded



"Risky" conclusion

$f(A) = 8 \cdot 2 = 16$

$x + 4y = 16$

$B(16, 0)$ belongs to the constraints

$f(B) = 16 \cdot 0 = 0 < f(A)$

so we are tempted to say that A is a maximum point.

In this case $A$ is actually a maximum but this way of thinking may not be correct.

$$\max \quad z = 2x^3 - y^3$$

$$\text{s.t.} \quad y = x$$

$$y - x = 0$$

$$h(x, y) = y - x$$

$$\nabla h = (-1, 1) \neq (0, 0)$$

$$\mathcal{L} = 2x^3 - y^3 - \mu(y - x)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 6x^2 + \mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = -3y^2 - \mu = 0$$

$$\begin{cases} 6x^2 + \mu = 0 \\ 3y^2 + \mu = 0 \\ y = x \end{cases} \qquad \begin{cases} 6x^2 + \mu = 0 \\ 3x^2 + \mu = 0 \\ y = x \end{cases} \qquad \begin{cases} 6x^2 - 3x^2 = 0 \\ \mu = -3x^2 \\ y = x \end{cases}$$

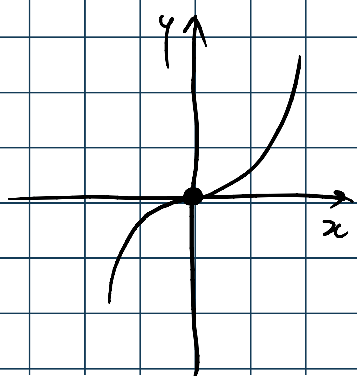
$$\begin{cases} x = 0 \\ \mu = 0 \\ y = 0 \end{cases} \qquad (0, 0)$$

$$\max \quad 2x^3 - y^3$$

$$\text{s.t.} \quad y = x$$

$$\downarrow$$

$$\max \; 2x^3 - x^3 \longrightarrow \max \; x^3$$



s.t. $y = x$

$$\max\ 2x^3 - x^3 \longrightarrow \max\ x^3$$