Theorem: let f and h be C^1 functions of two variables. Suppose that $\underline{x}^* = (x^*, y^*)$ is a solution of the problem

$$\max \quad f(x, y)$$

$$\text{s.t.:} \quad h(x, y) = c$$

Suppose further that (x^*, y^*) is not a critical point of h (that is $\nabla h(x^*, y^*) \neq (0, 0)$). Then, there is a real number μ^* such that (x^*, y^*, μ^*) is a critical point of the Lagrangian function

$$\mathcal{L}(x, y, \overset{\lambda}{\mu}) = f(x, y) - \mu\,(h(x, y) - c)$$

In other, at $(x^*, y^*, \overset{\lambda}{\mu^*})$

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \qquad \frac{\partial \mathcal{L}}{\partial y} = 0, \qquad \frac{\partial \mathcal{L}}{\partial \mu} = 0.$$

$$\max \quad f(x_1, x_2, \dots x_n)$$

$$\text{s.t.:} \quad h_1(x_1, x_2, \dots, x_n) = a_1$$

$$h_2(x_1, x_2, \dots, x_n) = a_2$$

$$\vdots \qquad\qquad \vdots$$

$$h_m(x_1, x_2, \dots, x_n) = a_m$$

constraint set C_h : it is the set of

constraint set C_h: it is the set of points that satisfy the constraints above

$$C_h = \{ \underline{x} \in \mathbb{R}^n : h_1(\underline{x}) = a_1,\ h_2(\underline{x}) = a_2,\ \dots,\ h_m(\underline{x}) = a_m \}$$

$$\cdots \quad \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix} = \underline{\underline{J}}_h(\underline{x}) = D_{\underline{h}}(\underline{x})$$

Jacobian matrix

non-degenerate constraint qualification (NDCQ):

rank $\underline{\underline{J}}_h(\underline{x}^*) = m$ at the critical point $\underline{x}^*$

$$\begin{cases} 2x - 2y = 8 \\ x + y = 2 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -2 & 8 \\ 1 & 1 & 2 \end{array}\right] \quad R_1 \leftarrow \tfrac{1}{2} R_1 \quad \sim$$

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 1 & 1 & 2 \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$$

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 2 & 6 \end{array}\right]$$ echelon form

pivot

$$\text{rank}\,(\underline{A} \mid \underline{b}) = 2$$

THEOREM: let $f, h_1, \dots, h_m$ be C^1 functions of n variables. Consider the problem of maximizing (minimizing) $f(\underline{x})$ on the constraint set C_h. Suppose that $\underline{x}^* \in C_h$ and that $\underline{x}^*$ is a local max (or min) of f on C_h. Suppose further that $\underline{x}^*$ satisfies the NDCQ condition. Then, there exist $\mu_1^*, \dots, \mu_m^*$ such that

$$(\underbrace{x_1^*, \dots, x_m^*}_{\underline{x}^*}, \underbrace{\mu_1^*, \dots, \mu_m^*}_{\underline{\mu}^*})$$ is a critical point of the Lagrangian

$$\mathcal{L}(\underline{x}, \underline{\mu}) = f(\underline{x}) - \mu_1 (h_1(\underline{x}) - a_1) - \dots - \mu_m (h_m(\underline{x}) - a_m)$$

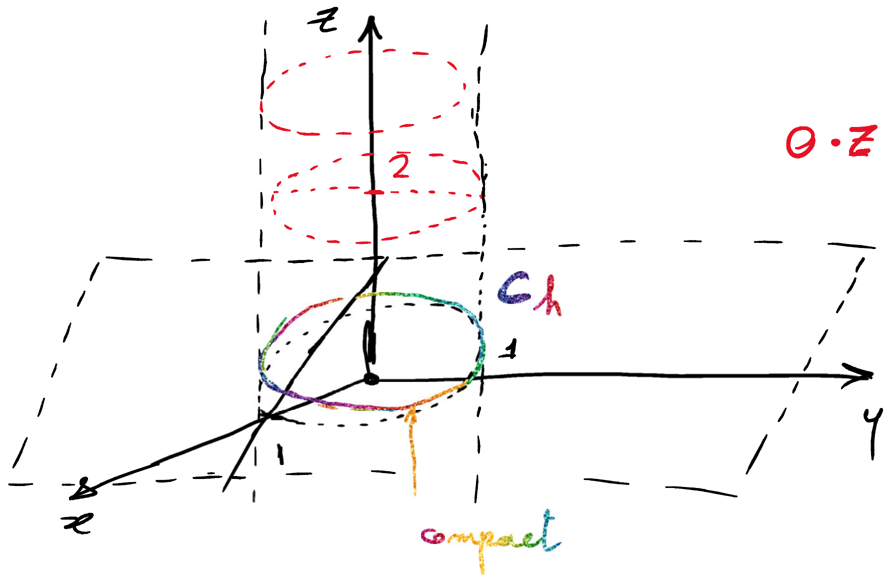
In other words,

$$\frac{\partial \mathcal{L}}{\partial x_1}(\underline{x}^*, \underline{\mu}^*) = 0, \quad \dots, \quad \frac{\partial \mathcal{L}}{\partial x_n}(\underline{x}^*, \underline{\mu}^*) = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \mu_1}(\underline{x}^*, \underline{\mu}^*) = 0, \quad \dots, \quad \frac{\partial \mathcal{L}}{\partial \mu_m}(\underline{x}^*, \underline{\mu}^*) = 0$$

$$\begin{aligned} \max \quad & xyz \\ \text{s.t.} \quad & x^2 + y^2 = 1 \longleftarrow \\ & x + z = 1 \end{aligned}$$

$$f(x, y, z) = xyz$$



$$0 \cdot z + x^2 + y^2 = 1$$

cylinder

$$x + z = 1$$
$$z = 1 - x$$

The constraint set is compact. The function $f(x, y, z)$ is continuous and so by Weierstraß's theorem there is a global maximum and minimum of f over C_h

$$x^2 + y^2 = 1 \qquad h_1(x,y,z) = x^2 + y^2$$

$$x + z = 1 \qquad h_2(x,y,z) = x + z$$

$$\underline{\underline{J}}_h = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial z} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2x & 2y & 0 \end{bmatrix}$$

$(0, 0, 1)$ $\quad (0, 0, \sqrt{2})$

$(0, 0, z)$

If $x = y = 0$ the second rows becomes

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = 0, \quad y = 1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

Points in the form $(0, 0, z)$ do not satisfy the constraint qualifications.
But these points are not feasible.
If we substitute their coordinates into

$x^2 + y^2 = 1$ we get $0^2 + 0^2 = 1$; $0 = 1$

that is impossible

| we get $0 + 0 = 1$; $0 = 1$

that is impossible.

This means that there are points that do not satisfy the constraint qualifications but they are not in C_h.

In other words, all the points in C_h satisfy the constraint qualification.

$$\mathcal{L}(x, y, z, \mu_1, \mu_2) =$$

$$= xyz - \mu_1 (x^2 + y^2 - 1) - \mu_2 (x + z - 1)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \qquad yz - 2\mu_1 x - \mu_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \qquad xz - 2\mu_1 y = 0$$

$$\frac{\partial \mathcal{L}}{\partial z} = 0 \qquad xy - \mu_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu_1} = 0 \qquad x^2 + y^2 = 1$$

$$\frac{\partial \mathcal{L}}{\partial \mu_2} = 0 \qquad x + z = 1$$

$$\begin{cases} yz - 2x\mu_1 - \mu_2 = 0 \\ xz - 2y\mu_1 = 0 \\ xy - \mu_2 = 0 \\ x^2 + y^2 = 1 \\ x + z = 1 \end{cases}$$

$$\begin{cases} x + y = 1 \\ x + z = 1 \end{cases}$$

$$xz = 2y\mu_1 \quad ; \qquad \text{either} \quad \text{(A)} \quad \mu_1 = \frac{xz}{2y} \quad \text{if } y \neq 0$$

$$\text{or} \quad y = 0 \quad \text{(B)}$$

(B)

$$\begin{cases} -2x\mu_1 - \mu_2 = 0 \\ y = 0 \\ -\mu_2 = 0 \\ x^2 = 1 \\ x + z = 1 \end{cases} \qquad \begin{cases} -2x\mu_1 = 0 \\ y = 0 \\ \mu_2 = 0 \\ x = \pm 1 \\ x + z = 1 \end{cases}$$

<u>B.1</u>

$$\begin{cases} \mu_1 = 0 \\ y = 0 \\ \mu_2 = 0 \\ x = 1 \\ z = 0 \end{cases} \qquad A(1, 0, 0)$$

<u>B.2</u>

$$\begin{cases} \mu_1 = 0 \\ y = 0 \\ \mu_2 = 0 \\ x = -1 \\ z = 2 \end{cases} \qquad B(-1, 0, 2)$$

(A)

$$\begin{cases} yz - 2x\mu_1 - \mu_2 = 0 \\ \mu_1 = \frac{xz}{2y} \\ xy \quad \mu \quad = 0 \end{cases}$$

$$\begin{cases} \qquad\qquad 2y \\ xy - \mu_2 = 0 \\ x^2 + y^2 = 1 \\ x + z = 1 \end{cases}$$

$$\begin{cases} yz - 2x \cdot \dfrac{xz}{2y} - xy = 0 \\ \mu_1 = \dfrac{xz}{2y} \\ \mu_2 = xy \\ x^2 + y^2 = 1 & \qquad y^2 = 1 - x^2 \\ x + z = 1 & \qquad z = 1 - x \end{cases}$$

$$yz - \cancel{2}x \cdot \frac{xz}{\cancel{2}y} - xy = 0$$

$$yz - \frac{x^2 z}{y} - xy = 0$$

$$y \cdot \frac{y^2 z - x^2 z - xy^2}{\cancel{y}} = 0 \cdot y$$

this is possible because now $y \neq 0$

$$z(y^2 - x^2) - xy^2 = 0$$

$$(1-x)(1-x^2-x^2) - x(1-x^2) = 0$$

$$(1-x)(1-2x^2) - x(1-x)(1+x) = 0$$

$$(1-x)\left[1 - 2x^2 - x(1+x)\right] = 0$$

$$(1-x)(1-2x^2-x-x^2)=0$$

$$(1-x)(1-x-3x^2)=0$$

$$(x-1)(3x^2+x-1)=0$$

$$x=1 \quad \vee \quad 3x^2+x-1=0$$

$$\Delta = 1-4\cdot 3\cdot(-1) = 1+12 = 13$$

$$x=1 \quad \vee \quad x_C = \frac{-1-\sqrt{13}}{6} \quad \vee \quad x_D = \frac{-1+\sqrt{13}}{6}$$

$$y_C = \pm\sqrt{1-x_C^2}$$

$$f(C) = f(D) = f(E) = f(F) \cong -0.8696$$

$$A(1,0,0) \qquad f(A) = 0$$

$$B(-1,0,2) \qquad f(B) = 0$$