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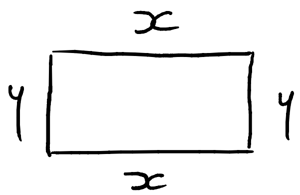
Find the rectangle of maximum area among those with perimeter equal to $2p$.



$$\underbrace{2x + 2y}_{\text{perimeter}} = 2p \longrightarrow x + y = p$$

$$\text{area} = xy$$

$$\max \quad xy$$
$$\text{s.t.} \quad x + y = p$$

$$f(x, y) = xy$$

$$h(x, y) = x + y$$

Constraint qualification: $\nabla h = (1, 1)^T \neq (0, 0)^T$

ok

$$\mathcal{L}(x, y, \mu) = xy - \lambda(x + y - p)$$

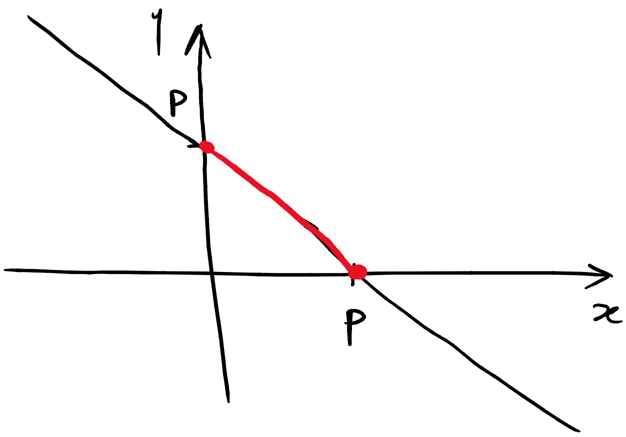
$$\frac{\partial \mathcal{L}}{\partial x} = y - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x + y - p) = 0$$

$$\begin{cases} y = \lambda \\ x = \lambda \\ x + y = P \end{cases} \qquad \begin{cases} y = \lambda \\ x = \lambda \\ 2\lambda = P \end{cases} \qquad \begin{cases} y = \frac{P}{2} \\ x = \frac{P}{2} \\ \lambda = \frac{P}{2} \end{cases}$$

The desired rectangle is a square!

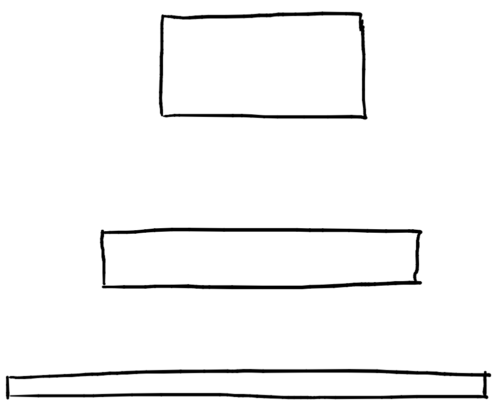


$x + y = P$

This line is a closed set but it is not bounded so we can't apply Weierstrass's theorem.

But, since a rectangle cannot have negative sides, we have implicitly assumed that $x \geq 0$ and $y \geq 0$

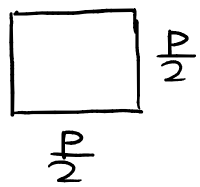
So the feasible set becomes the red segment that is closed and also bounded and so we can apply Weierstrass's theorem



degenerate case

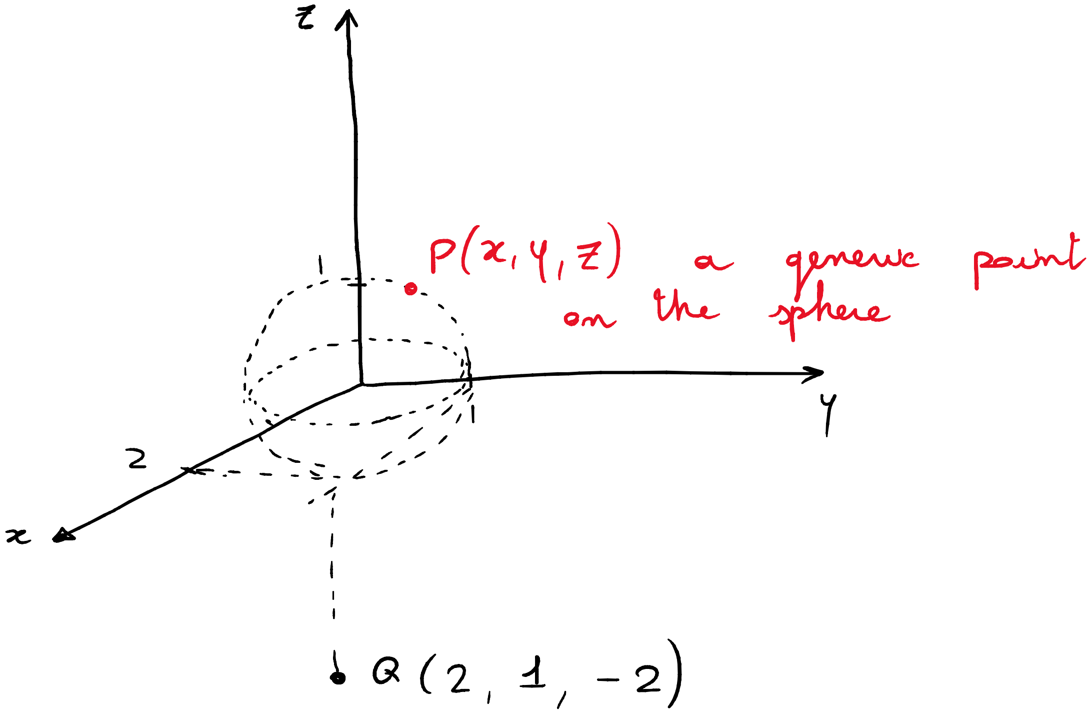
$x = 2p$ $\qquad y = 0$ $\qquad$ Area $= 0$

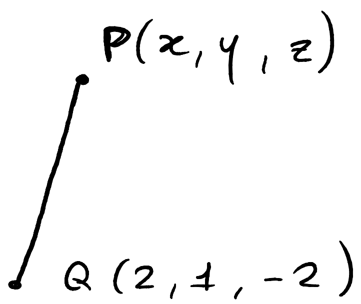
minimum area



maximum area

Find the minimum and maximum distance of the point $Q(2, 1, -2)$ from the sphere $x^2 + y^2 + z^2 = 1$. ←





$$d(P, Q) = \sqrt{(x-2)^2 + (y-1)^2 + (z+2)^2}$$

$$\max_{\text{or } \min} \sqrt{(x-2)^2 + (y-1)^2 + (z+2)^2}$$

s.t.: $\quad x^2 + y^2 + z^2 = 1$

If we consider $d^2(P, Q)$ instead of $d(P, Q)$ the minimum and maximum points do not change.

$$\max / \min \quad (x-2)^2 + (y-1)^2 + (z+2)^2$$

s.t.: $\quad x^2 + y^2 + z^2 = 1$

$$h(x, y, z) = x^2 + y^2 + z^2$$

$$\nabla h = (2x, 2y, 2z) \neq (0, 0, 0)$$

$(0, 0, 0)$ does not satisfy the constraint qualification but does not belong to the constraint set (i.e., the sphere) and so all the points on the sphere satisfy the constraint qualification.

$$\mathcal{L}(x, y, z, \lambda) = (x-2)^2 + (y-1)^2 + (z+2)^2 - \lambda(x^2 + y^2 + z^2 - 1)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2(x-2) - 2\lambda x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2(y-1) - 2\lambda y = 0$$

$$\frac{\partial \mathcal{L}}{\partial z} = 2(z+2) - 2\lambda z = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} \rightarrow \quad x^2 + y^2 + z^2 = 1$$

$$\begin{cases} x - 2 - \lambda x = 0 \\ y - 1 - \lambda y = 0 \\ z + 2 - \lambda z = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases} \qquad \begin{cases} (1-\lambda)\, x = 2 \\ (1-\lambda)\, y = 1 \\ (1-\lambda)\, z = -2 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

$$x = \frac{2}{1-\lambda} \quad \text{if} \quad \lambda \neq 1$$

if $\lambda = 1$ the system becomes

$$\begin{cases} 0 \cdot x = 2 \rightarrow 0 = 2 \\ \vdots \\ \end{cases}$$

impossible

$$\begin{cases} (1-\lambda)\, x = 2 \\ (1-\lambda)\, y = 1 \\ (1-\lambda)\, z = -2 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

$$\begin{cases} x = \dfrac{2}{1-\lambda} \\ y = \dfrac{1}{1-\lambda} \\ z = -\dfrac{2}{1-\lambda} \\ \left(\dfrac{2}{1-\lambda}\right)^2 + \left(\dfrac{1}{1-\lambda}\right)^2 + \left(-\dfrac{2}{1-\lambda}\right)^2 = 1 \end{cases}$$

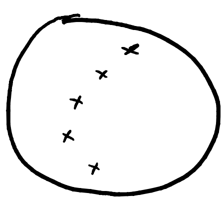
$$\frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 1$$

$$\frac{9}{(1-\lambda)^2} = 1 \quad ; \quad (1-\lambda)^2 = 9 \quad ; \quad (\lambda - 1)^2 = 9$$

$$\lambda - 1 = \pm 3 \qquad \lambda = 1 \pm 3 \begin{cases} 4 \\ -2 \end{cases}$$

$$\begin{cases} x = \dfrac{2}{1-\lambda} \\ y = \dfrac{1}{1-\lambda} \\ z = -\dfrac{2}{1-\lambda} \\ \left(\dfrac{2}{1-\lambda}\right)^2 + \left(\dfrac{1}{1-\lambda}\right)^2 + \left(-\dfrac{2}{1-\lambda}\right)^2 = 1 \end{cases} \qquad \begin{cases} x = -\dfrac{2}{3} \\ y = -\dfrac{1}{3} \\ z = \dfrac{2}{3} \\ \lambda = 4 \end{cases} \vee \begin{cases} x = \dfrac{2}{3} \\ y = \dfrac{1}{3} \\ z = -\dfrac{2}{3} \\ \lambda = -2 \end{cases}$$

$$A\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) \qquad B\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$



closed and bounded

By Weierstrass's theorem, there is a global minimum and a global maximum.

$$d(P, Q) = \sqrt{(x-2)^2 + (y-1)^2 + (z+2)^2}$$

$$A\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) \qquad B\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$

$$d(A, Q) = \sqrt{\left(-\frac{2}{3}-2\right)^2 + \left(-\frac{1}{3}-1\right)^2 + \left(\frac{2}{3}+2\right)^2} =$$

$$= \cdots = 4 \qquad \text{max}$$

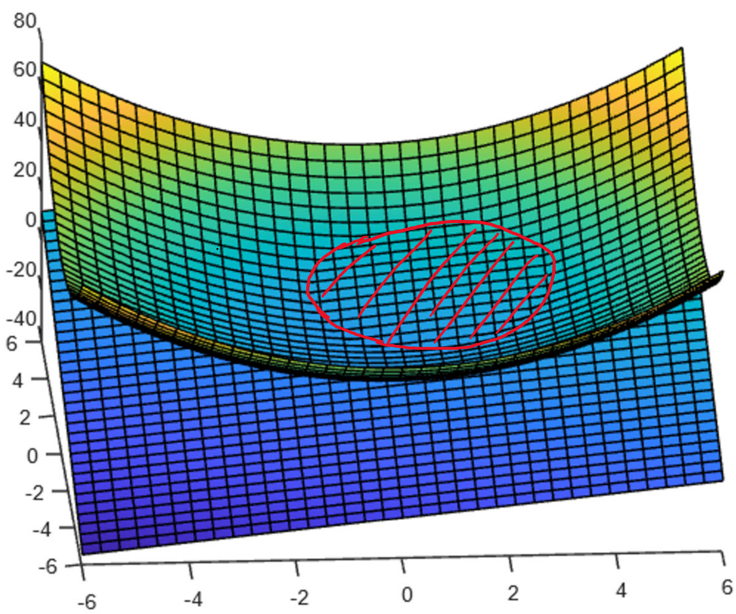
$$d(B, Q) = 2 \qquad \text{min.}$$

∞

$$\max \quad x + 3y - z$$

$$\text{s.t.:} \quad z = x^2 + y^2$$

$$2x + 4y - z = 0$$



closed and bounded

$$\max \quad x + 3y - z$$

$$\text{s.t.}: \quad z = x^2 + y^2$$

$$2x + 4y - z = 0$$

$$\max \quad x + 3y - z$$

$$\text{s.t.} \quad x^2 + y^2 - z = 0$$

$$2x + 4y - z = 0$$

$$h_1(x, y, z) = x^2 + y^2 - z$$

$$h_2(x, y, z) = 2x + 4y - z$$

$$\underline{\underline{J}} = \begin{bmatrix} 2x & 2y & -1 \\ 2 & 4 & -1 \end{bmatrix}$$

If $x = 1$ and $y = 2$ we get $\begin{bmatrix} 2 & 4 & -1 \\ 2 & 4 & -1 \end{bmatrix}$

$$\sim$$

$$\begin{cases} 2x + 4y - z = 0 \\ \cancel{2x + 4y - z = 0} \end{cases} \qquad \begin{bmatrix} 2 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

the rank is 1

So $(x, y) = (1, 2)$ does not satisfy NDQC

$$x^2 + y^2 - z = 0$$

$$x^2 + y^2 - z = 0$$

$$2x + 4y - z = 0$$

$$z = x^2 + y^2 = 1^2 + 2^2 = 5$$

$$z = 2x + 4y = 2 \cdot 1 + 4 \cdot 2 = 10$$

so $(x, y) = (1, 2)$ does not belong to the constraint set

$$\mathcal{L}(x, y, z, \lambda, \mu) = x + 3y - z - \lambda(x^2 + y^2 - z) - \mu(2x + 4y - z)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda x - 2\mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 3 - 2\lambda y - 4\mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial z} = -1 + \lambda + \mu = 0$$

$$z = x^2 + y^2$$

$$z = 2x + 4y$$

$$\begin{cases} 2\lambda x = 1 - 2\mu \\ 2\lambda y = 3 - 4\mu \\ \lambda + \mu = 1 \\ x^2 + y^2 = 2x + 4y \\ z = 2x + 4y \end{cases}$$

if $\lambda = 0$ $\quad \begin{cases} 0 = 1 - 2\mu \\ 0 = 3 - 4\mu \\ \mu = 1 \\ \vdots \end{cases} \quad \begin{cases} \mu = \frac{1}{2} \\ \mu = \frac{3}{4} \\ \mu = 1 \end{cases}$ impossible

$$\begin{cases} x = \dfrac{1 - 2\mu}{2\lambda} = \dfrac{1 - 2(1-\lambda)}{2\lambda} = \dfrac{-1 + 2\lambda}{2\lambda} = -\dfrac{1}{2\lambda} + 1 \\ y = \dfrac{3 - 4\mu}{\ldots} = \dfrac{3 - 4(1-\lambda)}{\ldots} = \dfrac{3 - 4 + 4\lambda}{\ldots} = -\dfrac{1}{\ldots} + 2 \end{cases}$$

$$\begin{cases} y = \dfrac{3-4\mu}{2\lambda} = \dfrac{3-4(1-\lambda)}{2\lambda} = \dfrac{3-4+4\lambda}{2\lambda} = -\dfrac{1}{2\lambda}+2 \\ \mu = 1-\lambda \\ x^2-2x+y^2-4y=0 \\ z = 2x+4y \end{cases}$$

$$\underbrace{x^2-2x+1}+y^2-4y+4 = 1+4$$

$$(x-1)^2 + (y-2)^2 = 5$$

$$\left(-\frac{1}{2\lambda}+\cancel{1-1}\right)^2+\left(-\frac{1}{2\lambda}+\cancel{2-2}\right)^2 = 5$$

$$\frac{1}{4\lambda^2}+\frac{1}{4\lambda^2} = 5 \quad ; \quad \frac{1}{2\lambda^2} = 5$$

$$\lambda^2 = \frac{1}{10} \quad ; \quad \lambda = \pm\frac{1}{\sqrt{10}}$$

$A\left(-\dfrac{\sqrt{10}}{2}+1,\ -\dfrac{\sqrt{10}}{2}+2,\ -3\sqrt{10}+10\right)$ global maximum

$B\left(\dfrac{\sqrt{10}}{2}+1,\ \dfrac{\sqrt{10}}{2}+2,\ 3\sqrt{10}+10\right)$ global minimum

$f(A) = \sqrt{10}-3$

$f(B) = -\sqrt{10}-3$