giovedì 20 aprile 2023 09:00

INEQUALITY CONSTRAINTS

max $f(x, y)$

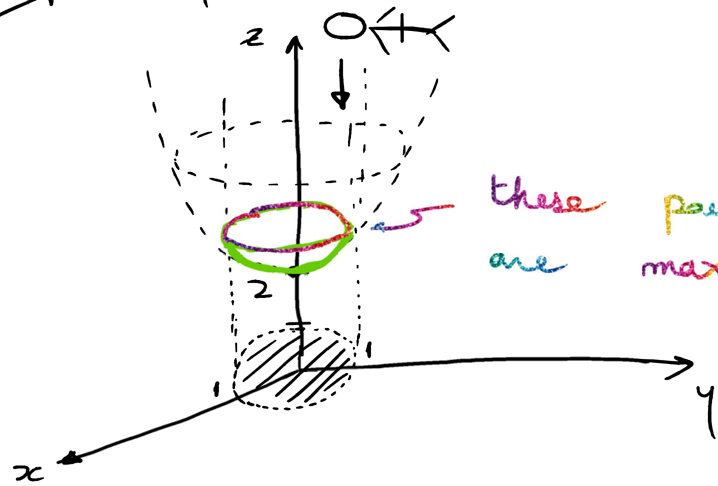
s.t.: $g(x,y) \leq b$

For example:

max | $x^2 + y^2 + 2 =: f(x,y)$

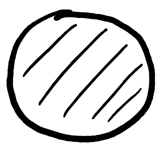
s.t.: $x^2 + y^2 \leq 1$

$z = x^2 + y^2$



these points on the border are maximum points

This means that there is at least one point (x_P, y_P) such that $x_P^2 + y_P^2 = 1$.



For example $(0, 1)$, $(-1, 0)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ satisfy the constraint $x^2 + y^2 \leq 1$ as an equality

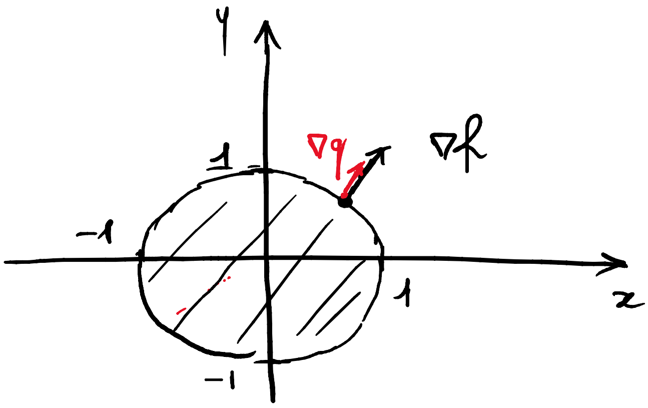
$1 \leq 1$

So the constraint is said to be active

on binding at (x_P, y_P).

$$f(x,y) = x^2 + y^2 \qquad \nabla f = (2x, 2y)^T$$

$$g(x,y) = x^2 + y^2 \qquad \nabla g = (2x, 2y)^T$$



Since the gradient tells us the direction of maximum growth and the maximum lies on the border, then ∇f has to point towards the exterior of the constraint set.

Otherwise, there would exist an interior point (x_Q, y_Q) such that $f(x_Q, y_Q) > f(\underline{x_P, y_P})$.

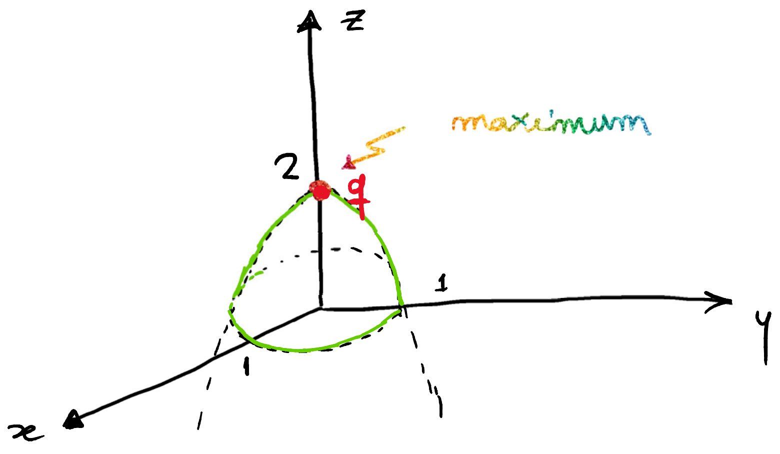
But also ∇g points outwards.

So, not only at $P(x_P, y_P)$ ∇f is parallel to ∇g (i.e., $\nabla f = \lambda \nabla g$) but they also have to have the same verse, that is $\lambda \geq 0$.

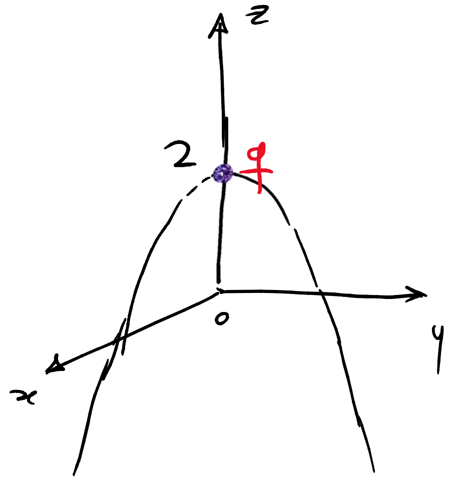
max $\quad 2 - x^2 - y^2$ | max $\quad 2 - x^2 - y^2$

$\max \quad 2 - x - y$

s.t.: $\quad x^2 + y^2 \leq 1$



$\max \quad 2 - x - y$



In this other example the maximum is inside the constraint set. It is denoted by point q. But point q is the same maximum of the unconstrained problem

$$\max \quad 2 - x^2 - y^2$$

$$\mathcal{L}(x, y, \lambda) = 2 - x^2 - y^2 - \lambda\left(x^2 + y^2 - 1\right)$$

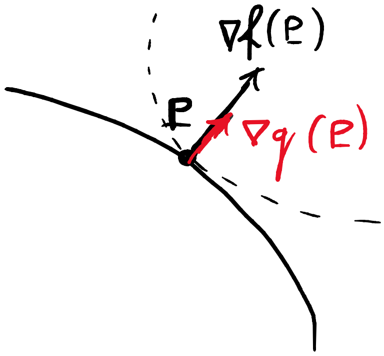
If $\lambda = 0$, $\quad \mathcal{L}(x, y, 0) = 2 - x^2 - y^2$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= 0 \end{aligned}\right\} \text{ is equivalent to } \quad \nabla f = \underline{0}$$

To summarize: form the Lagrangian:

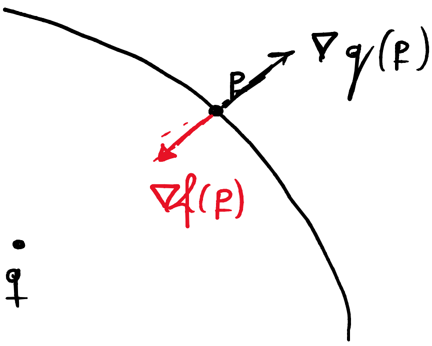
$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda\left(g(x, y) - b\right)$$

A) either the constraint is active



$\nabla f(P) \parallel \nabla g(P)$ and have the same verse
i.e. $\nabla f(P) = \lambda \nabla g(P)$ with $\lambda \geq 0$.

B) either the maximum is inside the constraint set.



In this case the constraint has no effect and so it must be $\lambda = 0$.

In both cases either $g(x,y) = b$ (case A) or $\lambda = 0$ (case B), that is either $g(x,y) \leq b$ or $\lambda \geq 0$ is active
This condition is called complementary slackness and can be expressed as:

$$\lambda\,[\,g(x,y) - b\,] = 0$$

<u>**THEOREM**</u>: suppose that f and g are C^1 functions on $\mathbb{R}^2$ and that (x^*, y^*) maximizes f on the constraint set $g(x,y) \leq b$.

If $g(x^*, y^*) = b$, suppose that

$$\frac{\partial g}{\partial x}(x^*, y^*) \neq 0 \quad \text{or} \quad \frac{\partial g}{\partial y}(x^*, y^*) \neq 0$$

(constraint qualification $\nabla g(x^*, y^*) \neq \underline{0}$)

In any case, form the Lagrangian

$$\mathcal{L}(x, y, \lambda) = f(x,y) - \lambda\,(g(x,y) - b)$$

Then, there is a multiplier λ^* such that:

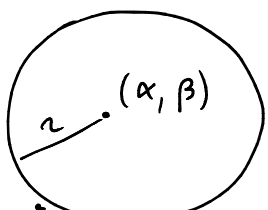
$$\frac{\partial \mathcal{L}}{\partial x}(x^*, y^*, \lambda^*) = 0, \quad \frac{\partial \mathcal{L}}{\partial y}(x^*, y^*, \lambda^*) = 0$$

$$\lambda^*(g(x^*, y^*) - b) = 0$$

$$\lambda^* \geq 0, \qquad g(x^*, y^*) \leq b.$$

These are necessary conditions.

circles: $(x-\alpha)^2 + (y-\beta)^2 = r^2$



lines: implicit form $ax + by + c = 0$ or explicit form $y = mx + q$, $x = k$

parabola: $y = ax^2 + bx + c$

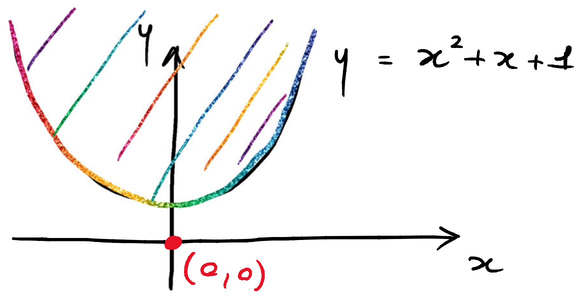
ellypse $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

hyperbola : $\quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad , \quad xy = K$

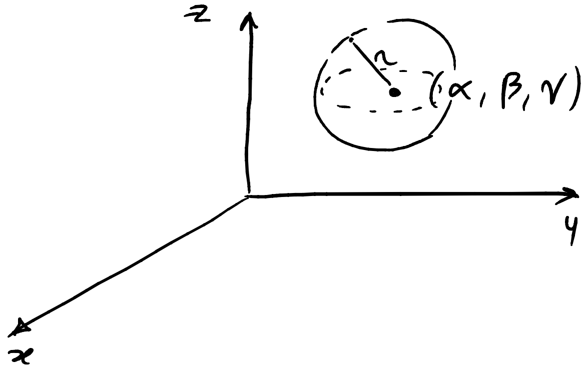
$y \geq x^2 + x + 1$

$0 \geq 0^2 + 0 + 1$

$0 \geq 1 \qquad$ NO



$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$$

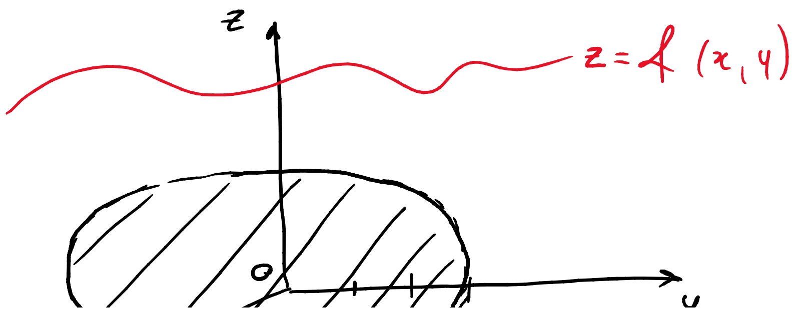


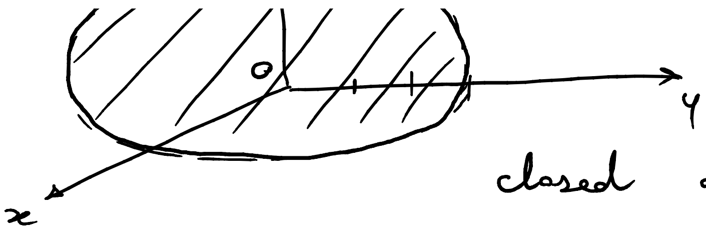
∞

max
min $\quad f(x,y) = x^4 + y^4 - 8(x^2+y^2)$

continuous

s.t. : $\quad x^2 + y^2 \leq 9$





closed and bounded

So by Weierstrass's theorem there is a global maximum and minimum.

$$\mathcal{L}(x,y,\lambda) = x^4+y^4-8(x^2+y^2)-\lambda(x^2+y^2-9)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} = 0 &: \\ \frac{\partial \mathcal{L}}{\partial y} = 0 &: \end{aligned} \begin{cases} \underline{4x^3-16x}-2\lambda x = 0 \\ \underline{4y^3-16y}-2\lambda y = 0 \\ \lambda(x^2+y^2-9) = 0 \quad \leftarrow \\ \lambda \geq 0 \\ x^2+y^2 \leq 9 \end{cases}$$

$\lambda = 0$

$4x^3-16x = 0$ and $4y^3-16y = 0$

$x^3-4x = 0$; $x(x^2-4) = 0$;

$x(x-2)(x+2) = 0$

$x = 0 \lor x = 2 \lor x = -2$

$y = 0 \lor y = 2 \lor y = -2$

✓ $A(0,0)$, $B(0,2)$, $C(0,-2)$, $D(2,0)$, $E(2,2)$, $F(2,-2)$, $G(-2,0)$, $H(-2,2)$, $I(-2,-2)$

$f'_x = 4x^3 - 16x = 0$ and $f'_y = 4y^3 - 16y = 0$

$f''_{xx} = 12x^2 - 16$ $\quad$ $f''_{xy} = 0$

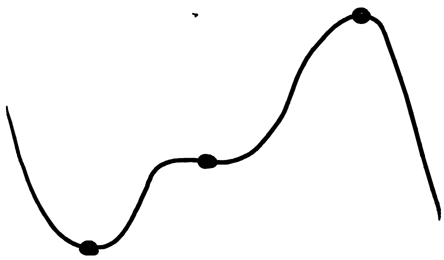
$f''_{yx} = 0$ $\quad$ $f''_{yy} = 12y^2 - 16$

$$\underline{\underline{H}}_f = \begin{bmatrix} 12x^2 - 16 & 0 \\ 0 & 12y^2 - 16 \end{bmatrix}$$

$$\underline{\underline{H}}_f(A) = \underline{\underline{H}}_f(0,0) = \begin{bmatrix} -16 & 0 \\ 0 & -16 \end{bmatrix}$$ A local maximum

$y = f(x)$

$f'(x) = 0$



$f'(x_p)$

$f''(x_p) < 0$

$$\underline{\underline{H}}_f(B) = \underline{\underline{H}}_f(C) = \underline{\underline{H}}_f(0, \pm 2) = \begin{bmatrix} -16 & 0 \\ 0 & 32 \end{bmatrix}$$ saddle points

also D and G

$$\underline{\underline{H}}_f(E) = \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}$$ local minima

same as F, H, I

$$\begin{cases} 4x^3 - 16x - 2\lambda x = 0 \\ 4y^3 - 16y - 2\lambda y = 0 \\ x^2 + y^2 - 9 = 0 \end{cases} \leftarrow$$

$$\begin{cases} 2x(2x^2 - 8\cancel{x} - \lambda) = 0 \\ 2y(2y^2 - 8\cancel{y} - \lambda) = 0 \\ x^2 + y^2 = 9 \end{cases}$$

$$\begin{cases} 2x(2x^2 - 8 - \lambda) = 0 \\ 2y(2y^2 - 8 - \lambda) = 0 \\ x^2 + y^2 = 9 \end{cases}$$

$$\begin{cases} x = 0 \\ \lambda = 10 \geq 0 \quad \checkmark \\ y = \pm 3 \end{cases} \qquad L(0, 3) \qquad M(0, -3)$$

$$\begin{cases} \lambda = 10 \geq 0 \quad \checkmark \\ y = 0 \\ x = \pm 3 \end{cases} \qquad N(3, 0) \qquad P(-3, 0)$$

$$\begin{cases} 2x^2 - 8 - \lambda = 0 \\ 2y^2 - 8 - \lambda = 0 \\ x^2 + y^2 = 9 \end{cases} \qquad \begin{cases} \lambda = 2x^2 - 8 \\ \lambda = 2y^2 - 8 \\ x^2 + y^2 = 9 \end{cases}$$

$$\begin{cases} \lambda = 2x^2 - 8 \\ 2x^2 - 8 = 2y^2 - 8 \\ x^2 + y^2 = 9 \end{cases} \qquad \begin{cases} \lambda = 2x^2 - 8 \\ x^2 = y^2 \\ 2x^2 = 9 \end{cases}$$

$$\begin{cases} \lambda = 1 \geq 0 \quad \checkmark \\ y = \pm \frac{3}{2}\sqrt{2} \\ x = \pm \frac{3}{2}\sqrt{2} \end{cases}$$

$Q\left(\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$, $R\left(\frac{3}{2}\sqrt{2}, -\frac{3}{2}\sqrt{2}\right)$,

$S\left(-\frac{3}{2}\sqrt{2},\ \frac{3}{2}\sqrt{2}\right),\quad T\left(-\frac{3}{2}\sqrt{2},\ -\frac{3}{2}\sqrt{2}\right)$

✓ $A(0,0)$, $B(0,2)$, $C(0,-2)$, $D(2,0)$,
$E(2,2)$, $F(2,-2)$, $G(-2,0)$, $H(-2,2)$,
$I(-2,-2)$

$L(0,3)$ $M(0,-3)$
$N(3,0)$ $P(-3,0)$

$Q\left(\frac{3}{2}\sqrt{2},\ \frac{3}{2}\sqrt{2}\right)$, $R\left(\frac{3}{2}\sqrt{2},\ -\frac{3}{2}\sqrt{2}\right)$,

$S\left(-\frac{3}{2}\sqrt{2},\ \frac{3}{2}\sqrt{2}\right)$, $T\left(-\frac{3}{2}\sqrt{2},\ -\frac{3}{2}\sqrt{2}\right)$

$$f(x,y) = x^4 + y^4 - 8(x^2+y^2)$$

$$f(A) = f(0,0) = 0$$

$$f(B) = f(C) = f(D) = f(G) = 2^4 - 8\cdot 2^2 = 2^4 - 2^5 =$$
$$= 2^4(1-2) = -16$$

$$f(E) = f(F) = f(H) = f(I) = 2^4 + 2^4 - 8(2^2+2^2) =$$
$$= 2^5 - 8\cdot 2^3 = 2^5 - 2^6 = -2^5 = -32$$

$$f(L) = f(M) = f(N) = f(P) = 3^4 - 8\cdot 3^2 =$$
$$= 3^2(3^2-8) = 3^2(9-8) = 3^2\cdot 1 = 9$$

$$f(Q) = f(R) = f(S) = f(T) =$$

$$= \left(\frac{3}{2}\sqrt{2}\right)^4 + \left(\frac{3}{2}\sqrt{2}\right)^4 - 8\left[\left(\frac{3}{2}\sqrt{2}\right)^2 + \left(\frac{3}{2}\sqrt{2}\right)^2\right] =$$

$$= \frac{3^4 \cdot 2^2}{2^4} + \frac{3^4 \cdot 2^2}{2^4} - 8\left(\frac{3^2 \cdot 2}{2^2} + \frac{3^2 \cdot 2}{2^2}\right) =$$

$$= 2 \cdot \frac{3^4}{2^2} - 8 \cdot \cancel{2} \cdot \frac{3^2}{\cancel{2}} = \frac{3^4}{2} - 8 \cdot 3^2 =$$

$$= 3^2\left(\frac{3^2}{2} - 8\right) = 9\left(\frac{9}{2} - 8\right) = 9 \cdot \frac{9 - 16}{2} =$$

$$= 9\left(-\frac{7}{2}\right) = -\frac{63}{2} = -\frac{62+1}{2} =$$

$$= -31 - \frac{1}{2} = -31.5$$