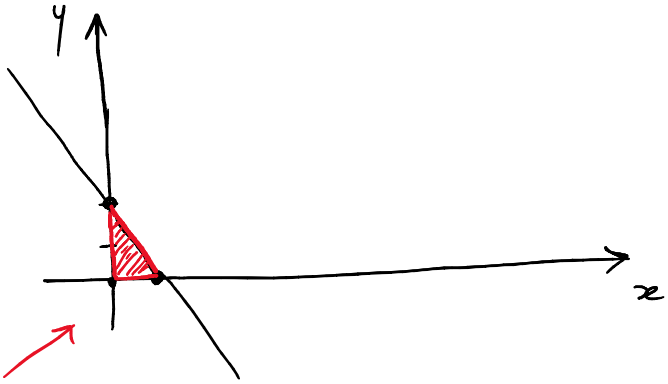
mercoledì 26 aprile 2023 09:02

$$\max \ f(x,y) = x^2 + y^2$$ continuous

s.t.:
$$2x + y \leqslant 2$$
$$x \geqslant 0$$
$$y \geqslant 0$$



constraint set
- closed
- boundet

} compact

$2x + y = 2$ border

If $x = 0$, $y = 2$

If $y = 0$, $x = 1$

$2x + y \leqslant 2$

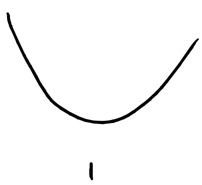
$(0,0)$ test point $\notin$ line

$2 \cdot 0 + 0 \leqslant 2$

$0 \leqslant 2$

By Weierstrass's theorem we have absolute maximum and minimum.

$$2x + y \leqslant 2$$
$$x \geqslant 0$$
$$(-1)\ x \leqslant 0\ (-1)$$
$$-x \leqslant 0$$

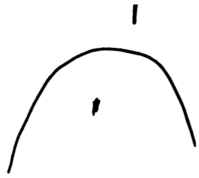


$$\max \quad x^2 + y^2$$
$$\text{s.t.} \quad 2x + y \leqslant 2$$
$$-x \leqslant 0$$

$\min \ g(x)$

$\max \ -g(x)$

$$-x \leqslant 0$$
$$-y \leqslant 0$$



$$g_1(x,y) = 2x + y$$
$$g_2(x,y) = -x$$
$$g_3(x,y) = -y$$

$$\underline{\underline{J}} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad R_2 \leftarrow 2R_2 + R_1$$

$$\begin{cases} 2x + y = \dots \\ -x = \dots \\ -y = \dots \end{cases} \qquad R_2 \leftarrow 2R_2$$

Gaussian method elimination

$$\begin{cases} 2x + y = \dots \\ -2x = 2\dots \\ -y = \dots \end{cases} \qquad R_2 \leftarrow R_2 + R_1$$

$$\begin{cases} 2x + y = \dots \\ 0 \quad y = \dots \\ 0 \quad -y = \dots \end{cases}$$

$$\underline{\underline{J}} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad R_2 \leftarrow 2R_2 \quad \sim$$

$$\begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 0 & -1 \end{bmatrix} \quad R_2 \leftarrow R_2 + R_1 \quad \sim$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad R_3 \leftarrow R_3 + R_2 \quad \sim$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

pivots

rank : number of pivots

rank = 2

$$\mathcal{L} = x^2 + y^2 - \lambda_1(2x + y - 2) - \lambda_2(-x) - \lambda_3(-y)$$

$$\mathcal{L} = x^2 + y^2 - \lambda_1(2x + y - 2) + \lambda_2 x + \lambda_3 y$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2x - 2\lambda_1 + \lambda_2$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y - \lambda_1 + \lambda_3$$

$$\left\{ \begin{array}{l} 2x - 2\lambda_1 + \lambda_2 = 0 \\ 2y - \lambda_1 + \lambda_3 = 0 \\ \lambda_1(2x + y - 2) = 0 \quad \leftarrow \end{array} \right.$$

$$\begin{cases} \lambda_1 (2x + y - 2) = 0 \quad \leftarrow \\ \lambda_2 x = 0 \\ \lambda_3 y = 0 \\ \lambda_1 \geq 0 \\ \lambda_2 \geq 0 \\ \lambda_3 \geq 0 \\ 2x + y \leq 2 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

① $\underline{\lambda_1 = 0}$

$$\begin{cases} 2x + \lambda_2 = 0 \\ 2y + \lambda_3 = 0 \\ \lambda_2 x = 0 \\ \lambda_3 y = 0 \end{cases} \qquad \begin{cases} \lambda_2 = -2x \\ \lambda_3 = -2y \\ -2x^2 = 0 \\ -2y^2 = 0 \end{cases} \qquad \begin{cases} \lambda_2 = 0 \geq 0 \quad \checkmark \\ \lambda_3 = 0 \geq 0 \quad \checkmark \\ x = 0 \\ y = 0 \end{cases}$$

$$\begin{matrix} 2x + y \leq 2 & & 0 \leq 2 & \checkmark \\ x \geq 0 & & 0 \geq 0 & \checkmark \\ y \geq 0 & & 0 \geq 0 & \checkmark \end{matrix}$$

$A(0, 0)$

2) $2x + y - 2 = 0$

$$\begin{cases} 2x - 2\lambda_1 + \lambda_2 = 0 \\ 2y - \lambda_1 + \lambda_3 = 0 \\ 2x + y - 2 = 0 \\ \lambda_2 x = 0 \quad \leftarrow \\ \lambda_3 y = 0 \end{cases}$$

$$\lambda_3 y = 0$$

2.1 $x = 0$

$$\begin{cases} -2\lambda_1 + \lambda_2 = 0 \\ 2y - \lambda_1 + \lambda_3 = 0 \\ y - 2 = 0 \\ \lambda_3 y = 0 \end{cases} \qquad \begin{cases} -2\lambda_1 + \lambda_2 = 0 \\ 4 - \lambda_1 + \lambda_3 = 0 \\ y = 2 \\ 2\lambda_3 = 0 \end{cases}$$

$$\begin{cases} \lambda_2 = 8 \geq 0 \ \checkmark \\ \lambda_1 = 4 \geq 0 \ \checkmark \\ y = 2 \\ \lambda_3 = 0 \quad \geq 0 \ \checkmark \end{cases}$$

$B(0, 2)$

$$\begin{aligned} 2x + y &\leq 2 & \qquad 2 &\leq 2 \ \checkmark \\ x &\geq 0 & 0 &\geq 0 \ \checkmark \\ y &\geq 0 & 2 &\geq 0 \ \checkmark \end{aligned}$$

2.2 $\lambda_2 = 0$

$$\begin{cases} 2x - 2\lambda_1 + \cancel{\lambda_2} = 0 \\ 2y - \lambda_1 + \lambda_3 = 0 \\ 2x + y - 2 = 0 \\ \lambda_2 = 0 \\ \lambda_3 y = 0 \end{cases}$$

2.2.1 $\lambda_3 = 0$

$$\begin{cases} 2x - 2\lambda_1 = 0 \\ 2y - \lambda_1 = 0 \\ 2x + y - 2 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases} \qquad \begin{cases} x = \lambda_1 \\ y = \frac{\lambda_1}{2} \\ 2\lambda_1 + \frac{\lambda_1}{2} = 2 \ ; \quad \frac{5}{2}\lambda_1 = 2 \ ; \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases}$$

$$\begin{cases} x = \frac{4}{5} \geq 0 \\ y = \frac{2}{5} \geq 0 \\ \lambda_1 = \frac{4}{5} \geq 0 \ \checkmark \\ \lambda_2 = 0 \geq 0 \ \checkmark \\ \lambda_3 = 0 \geq 0 \ \checkmark \end{cases}$$

$2x + y \leq 2$ ✓

$C\left(\frac{4}{5}, \frac{2}{5}\right)$

<u>2.2.2</u> $y = 0$

$$\begin{cases} 2x - 2\lambda_1 = 0 \\ -\lambda_1 + \lambda_3 = 0 \\ 2x - 2 = 0 \\ \lambda_2 = 0 \\ y = 0 \end{cases} \qquad \begin{cases} \lambda_1 = 1 \geq 0 \\ \lambda_3 = 1 \geq 0 \\ x = 1 \geq 0 \\ \lambda_2 = 0 \geq 0 \\ y = 0 \geq 0 \end{cases}$$ ✓

$D(1, 0)$

$f(x, y) = x^2 + y^2$

$A(0, 0)$ $\quad f(A) = 0$

$B(0, 2)$ $\quad f(B) = f(0, 2) = 4$

$C\left(\frac{4}{5}, \frac{2}{5}\right)$ $\quad f(C) = \frac{16}{25} + \frac{4}{25} = \frac{20}{25} = \frac{4}{5}$

$D(1, 0)$ $\quad f(D) = 1$

A: global minimum $\qquad$ B: global maximum

<u>Principal minors of a matrix</u>

Let $\underline{\underline{A}}$ be an $n \times n$ matrix. A $k \times k$ submatrix of $\underline{\underline{A}}$ formed by deleting $n - k$ columns and the same $n - k$ rows from $\underline{\underline{A}}$

columns and the same $n-k$ rows from $\underline{\underline{A}}$ is called a <u>k-th order principal submatrix</u> of $\underline{\underline{A}}$. The determinant of a $k \times k$ principal submatrix is called a k-th order principal minor of $\underline{\underline{A}}$.

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix} \qquad n = 3$$

3rd order principal submatrix $(k = 3)$

I need to delete $n - k$ columns and the same rows

$$3 - 3 = 0$$

The 3rd order principal submatrix is the matrix $\underline{\underline{A}}$ itself. leading

3rd order principal minor is $\det \underline{\underline{A}}$. leading

2nd order principal submatrix $(k = 2)$

We delete $n - k = 3 - 2 = 1$ column and the same row.

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$ 2nd order principal submatrix. $$\det \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = 9$$ 2nd order principal minor

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} \quad " \qquad \det \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = -2 \quad "$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \;\; \text{"} \qquad \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3 \;\; \text{"}$$

leading

1-st order principal submatrix $\qquad k = 1$

We delete $n - k = 3 - 1 = 2$ rows and the same columns from the matrix $\underline{\underline{A}}$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix} \qquad [4]$$

1 st order principal submatrix of $\underline{\underline{A}}$ $\qquad 4$ $\qquad$ 1 st order principal minor of $\underline{\underline{A}}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix} \qquad [2] \quad \text{"} \qquad 2 \quad \text{"}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & -1 & 4 \end{bmatrix} \qquad [0] \quad \text{"} \qquad 0 \quad \text{"}$$

leading

Let $\underline{\underline{A}}$ be an $n \times n$ matrix. The k-th order principal submatrix of $\underline{\underline{A}}$ obtained by deleting the last $n - k$ rows and the last $n - k$ columns from $\underline{\underline{A}}$ is called the k-th order leading principal submatrix of $\underline{\underline{A}}$. Its determinant is called the k-th order leading principal minor of $\underline{\underline{A}}$.

$$\begin{bmatrix} [1] & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$