giovedì 27 aprile 2023 09:03

$$\underline{A} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & -1 & -2 & 3 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

leading principal minors

$$|A_1| = 1$$

$$|A_2| = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -5$$

$$|A_3| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 0 & -1 & -2 \end{vmatrix}$$

$$|A_4| = |A|$$

$$Q(x_1, x_2) = [x_1 \quad x_2] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= a x_1^2 + 2b x_1 x_2 + c x_2^2$$

complete the square

$$\sim$$

$$(A + B)^2 = A^2 + 2AB + B^2$$

$$x^2 + x$$

you want to write is as a square plus something else

$$x^2 + x = \underbrace{x^2}_{A^2} + \underbrace{2 \cdot x \cdot \tfrac{1}{2}}_{2AB} + \underbrace{\tfrac{1}{4}}_{B^2} - \frac{1}{4} = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$$

$$A = x, \quad B = \frac{1}{2}$$

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

$$a^2x^2 + ax \cdot b + ac = 0$$

$$a^2x^2 + 2\underbrace{ax}\frac{b}{2} + ac = 0$$

$$A^2 + 2\,A\,B$$

$$\underbrace{a^2x^2 + 2ax\frac{b}{2} + \frac{b^2}{4}} + ac = \frac{b^2}{4}$$

$$\left(ax + \frac{b}{2}\right)^2 = \frac{b^2}{4} - ac$$

$$\left(ax + \frac{b}{2}\right)^2 = \frac{b^2 - 4ac}{4}$$

$$ax + \frac{b}{2} = \pm \frac{\sqrt{b^2 - 4ac}}{2}$$

...

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a \neq 0$$

$$Q(x_1, x_2) = ax_1^2 + 2b\,x_1x_2 + c\,x_2^2 =$$

$$= a\left(x_1^2 + 2\frac{b}{a}x_1x_2 + \frac{c}{a}x_2^2\right)$$

$$A^2 + 2 \quad A$$

$$B = \frac{b}{a}x_2$$

$$Q(x_1, x_2) = a\left(\underbrace{x_1^2 + 2x_1 \frac{b}{a} x_2 + \frac{b^2}{a^2} x_2^2} - \frac{b^2}{a^2} x_2^2 + \frac{c}{a} x_2^2\right) =$$

$$= a\left[\left(x_1 + \frac{b}{a} x_2\right)^2 + \frac{ac - b^2}{a^2}\, x_2^2\right] =$$

$$= \underline{a}\,\underbrace{\left(x_1 + \frac{b}{a} x_2\right)^2}_{\geqslant 0} + \underline{\frac{ac - b^2}{a}}\,\underbrace{x_2^2}_{\geqslant 0}$$

So, if $a > 0$ and $ac - b^2 > 0$, then Q is positive definite

If $a < 0$ and $ac - b^2 < 0$, then Q is negative definite

$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

1-st order leading principal minor : a

2-nd " " " " : $ac - b^2$

$$Q(x_1, x_2) = a\left(x_1 + \frac{b}{a} x_2\right)^2 + \frac{ac - b^2}{a}\, x_2^2$$

Let's see what happens if $a > 0$ and $ac - b^2 < 0$

$$Q\left(\frac{b}{a},-1\right)=\underbrace{a\cancel{\left(\frac{b}{a}+\frac{b}{a}(-1)\right)^2}}+\frac{ac-b^2}{a}(-1)^2=$$

$$=\frac{\overset{<0}{ac-b^2}}{\underset{>0}{a}}<0$$

$$Q(1,0)=a\left(1+\cancel{\frac{b}{a}0}\right)^2+\cancel{\frac{ac-b^2}{a}}0^2=a>0$$

In this case, the quadratic form is indefinite.

THEOREM: let $\underline{\underline{A}}$ be an $n\times n$ symmetric matrix. Then:

a) $\underline{\underline{A}}$ is positive definite if and only if its n leading principal minors are positive

b) $\underline{\underline{A}}$ is negative definite if and only if its n leading principal minors alternate in sign as follows:

$|A_1|<0$, $|A_2|>0$, $|A_3|<0$, ... etc...

(or $|A_k|$ has the same sign as $(-1)^k$)

c) If some k-th order leading principal minor of $\underline{\underline{A}}$ is nonzero but does not fit either of the above two sign patterns, then $\underline{\underline{A}}$ is indefinite.

is indefinite.

THEOREM: let $\underline{A}$ be a $n \times n$ symmetric matrix. Then, $\underline{A}$ is positive semidefinite if and only if every principal minor of $\underline{A}$ is ≥ 0. $\underline{A}$ is negative semidefinite if and only if every principal minor of odd order is ≤ 0 and every principal minor of even order is ≥ 0.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

$|A_1| = 1 > 0$

$|A_2| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = 3 > 0$

$$|A_3| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} =$$

$$= -4 - 1 - 2(8-3) + 3(2+3) =$$

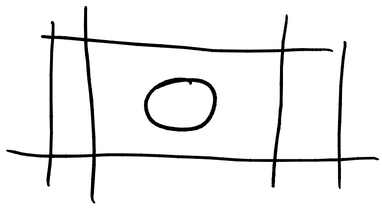
$$= -5 - 10 + 15 = 0$$

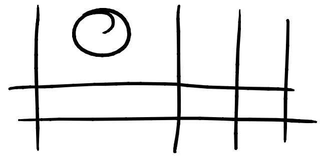
It is not positive definit nor negative definite

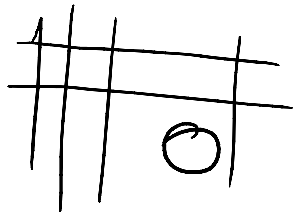
It is not positive semidefinite because one 1-st order principal minor is -1

It is neither negative semidefinite because one 1-st order principal minor is $1 > 0$

So our matrix is indefinite.







$$f(x,y) = (8x^2 - 6xy + 3y^2)\, e^{2x+3y}$$

$$\operatorname{dom} f = \mathbb{R}^2$$

$$f'_x = (16x - 6y)\, e^{2x+3y} +$$

$$+ (8x^2 - 6xy + 3y^2)\, 2\, e^{2x+3y} =$$

$$= 2(8x - 3y)\, e^{2x+3y} +$$

$$+ 2(8x^2 - 6xy + 3y^2)\, e^{2x+3y} =$$

$$= 2e^{2x+3y}\,(8x^2 - 6xy + 3y^2 + 8x - 3y)$$

$$f(x,y) = (8x^2 - 6xy + 3y^2)\, e^{2x+3y}$$

$$f'_y = (-6x + 6y)\, e^{2x+3y} +$$

$$+ (8x^2 - 6xy + 3y^2)\, 3\, e^{2x+3y} =$$

$$= 3(-2x + 2y)\, e^{2x+3y} \quad +$$

$$+ (8x^2 - 6xy + 3y^2)\, 3e^{2x+3y} \quad =$$

$$= 3(8x^2 - 6xy + 3y^2 - 2x + 2y)\, e^{2x+3y}$$

$$\nabla f = \underline{0}$$

$$\begin{cases} 2e^{2x+3y}\,(8x^2 - 6xy + 3y^2 + 8x - 3y) = 0 \\ 3(8x^2 - 6xy + 3y^2 - 2x + 2y)\, e^{2x+3y} = 0 \end{cases}$$

$$\begin{cases} 8x^2 - 6xy + 3y^2 + 8x - 3y = 0 \\ 8x^2 - 6xy + 3y^2 - 2x + 2y = 0 \end{cases} \qquad E_2 \leftarrow E_2 - E_1$$

$$\begin{cases} 8x^2 - 6xy + 3y^2 + 8x - 3y = 0 \\ \quad // \qquad // \qquad // \quad -10x + 5y = 0 \end{cases}$$

$$\begin{cases} 8x^2 - \cancel{12x^2} + \cancel{12x^2} + 8x - 6x = 0 \\ y = 2x \end{cases}$$

$$\begin{cases} 8x^2 + 2x = 0 \\ y = 2x \end{cases} \qquad \begin{cases} 4x^2 + x = 0 \\ y = 2x \end{cases}$$

$$\begin{cases} x(4x+1) = 0 \\ y = 2x \end{cases}$$

$$\begin{cases} x(4x+1) = 0 \\ y = 2x \end{cases}$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \quad \vee \quad \begin{cases} x = -\frac{1}{4} \\ y = -\frac{1}{2} \end{cases}$$

$$P(0, 0) \qquad Q\left(-\frac{1}{4}, -\frac{1}{2}\right)$$

$$f'_x = 2e^{2x+3y}\left(8x^2 - 6xy + 3y^2 + 8x - 3y\right)$$

$$f''_{xx} = 2 \cdot e^{2x+3y} \cdot 2\left(8x^2 - 6xy + 3y^2 + 8x - 3y\right) + $$
$$+ 2\underline{e^{2x+3y}}\left(16x - 6y + 8\right) =$$

$$= 4e^{2x+3y}\left(8x^2 - 6xy + 3y^2 + \underline{8x} - \underline{\underline{3y}}\right) +$$
$$+ 4e^{2x+3y}\left(\underline{8x} - \underline{\underline{3y}} + 4\right) =$$

$$= 4e^{2x+3y}\left(8x^2 - 6xy + 3y^2 + 16x - 6y + 4\right)$$

$$f''_{xy} = f''_{yx} = 6e^{2x+3y}\left(8x^2 - 6xy + 3y^2 + 6x - y - 1\right)$$

$$f''_{yy} = 3\left(24x^2 - 18xy + 9y^2 - 12x + 12y + 2\right)e^{2x+3y}$$

$P(0,0)$

$f''_{xx}(0,0) = 16$

$f''_{xy}(P) = f''_{yx}(P) = 6 \cdot 1 \, (-1) = -6$

$f''_{yy}(P) = 3 \cdot 2 \cdot 1 = 6$

$$\underline{\underline{H}}_f(P) = \begin{bmatrix} 16 & -6 \\ -6 & 6 \end{bmatrix}$$

$|H_1| = 16 > 0$

$|H_2| = \det \begin{bmatrix} 16 & -6 \\ -6 & 6 \end{bmatrix} = 16 \cdot 6 - 6^2 > 0$

positive definite $\qquad$ P is a local minimum

$$\underline{\underline{H}}_f = \begin{bmatrix} 14e^{-2} & -9e^{-2} \\ -9e^{-2} & \frac{3}{2}e^{-2} \end{bmatrix}$$

$|H_1| = \underbrace{14}_{>0}\underbrace{e^{-2}}_{>0} > 0$

$\det \underline{\underline{H}}_f = 14 \cdot e^{-2} \cdot \frac{3}{2} e^{-2} - 81 e^{-4} =$

$= e^{-4}\left(\cancel{14}^{7} \cdot \frac{3}{\cancel{2}} - 81\right) = -60\underbrace{e^{-4}}_{>0} < 0$

indefinite

Q is a saddle point.