Find the domain of the function

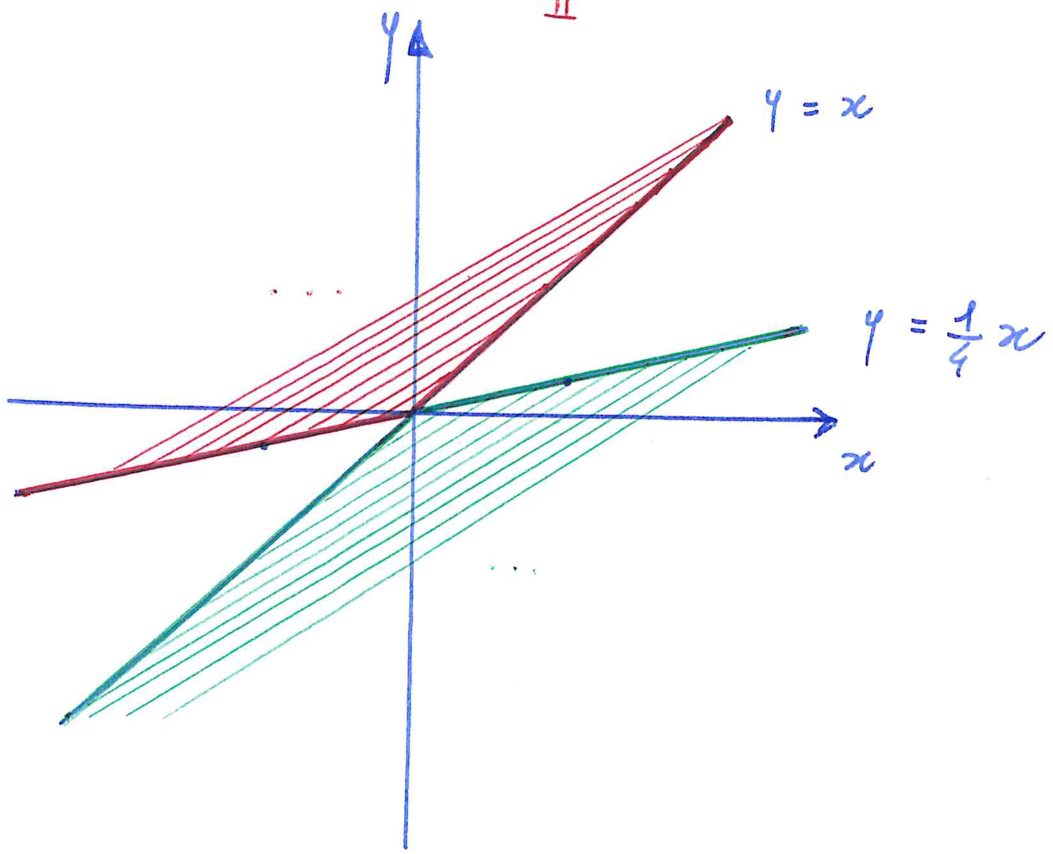
$$z = \sqrt{x^2 - 5xy + 4y^2}$$

$$x^2 - 5xy + 4y^2 \geqslant 0$$

$$(x-y)(x-4y) \geqslant 0$$

$$\begin{cases} x - y \geqslant 0 \\ x - 4y \geqslant 0 \end{cases} \quad \vee \quad \begin{cases} x - y \leqslant 0 \\ x - 4y \leqslant 0 \end{cases}$$

$$\underset{\mathrm{I}}{\begin{cases} y \leqslant x \\ y \leqslant \frac{1}{4}x \end{cases}} \quad \vee \quad \underset{\mathrm{II}}{\begin{cases} y \geqslant x \\ y \geqslant \frac{1}{4}x \end{cases}}$$



Classify the following quadratic form:

$$Q(x, y, z) = 5x^2 - y^2 + z^2 + 4xy + 6xz$$

$$Q(x, y, z) = \underline{x}^T \underline{\underline{A}}\, \underline{x}$$

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \underline{\underline{A}} = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

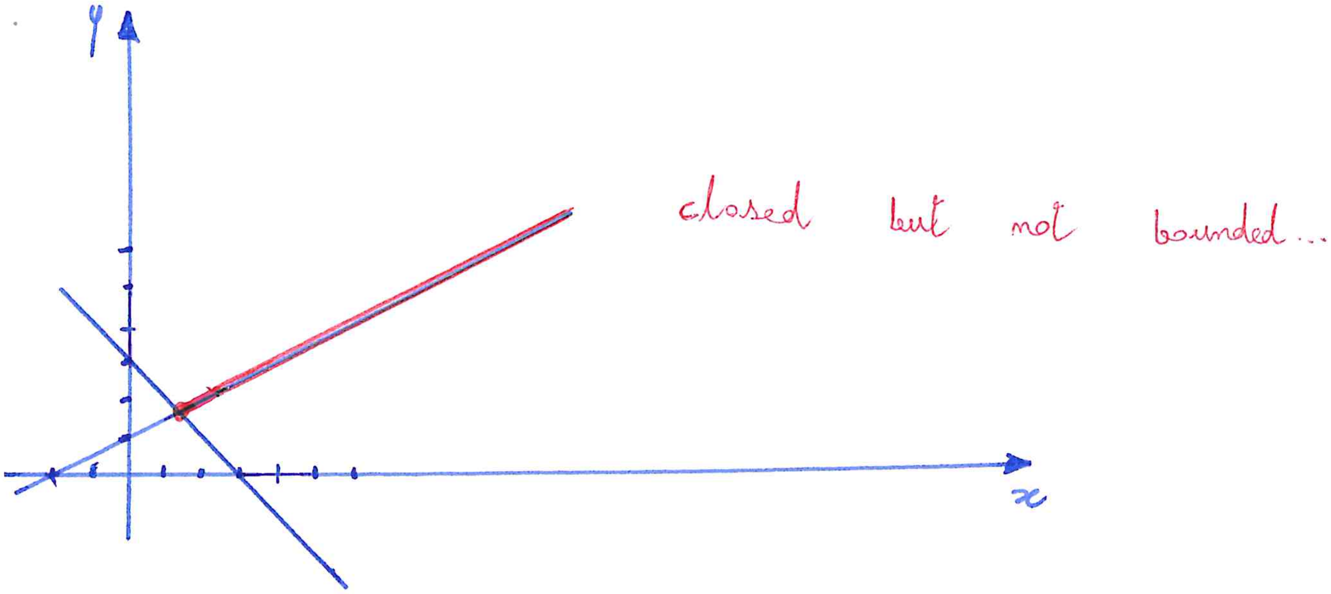
$$|A_1| = 5 > 0$$

$$|A_2| = \begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix} = -5 - 4 = -9$$

$$|A_3| = \begin{vmatrix} 5 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix} =$$

$$= 3 \cdot 3 + (-5 - 4) = 0$$

$\underline{\underline{A}}$ could be positive or negative semidefinite
But the first-order principal minors 5, -1, and 1 have different sign. Therefore, $\underline{\underline{A}}$ can be neither positive nor negative semidefinite.
Thus, $\underline{\underline{A}}$ is indefinite and the point $(0, 0, 0)$ is a saddle point.



$$\max \quad 3xy - x^3$$
s.t.:
$$x - 2y + 2 = 0$$
$$x + y \geq 3$$
$$x \geq 0$$
$$y \geq 0$$

$$\max \quad 3xy - x^3$$
s.t.:
$$x - 2y + 2 = 0$$
$$-x - y + 3 \leq 0$$
$$-x \leq 0$$
$$-y \leq 0$$

$$\mathcal{L} = 3xy - x^3 - \mu(x - 2y + 2) - \lambda_1(-x - y + 3) - \lambda_2(-x) - \lambda_3(-y)$$

$$\mathcal{L} = 3xy - x^3 - \mu(x - 2y + 2) + \lambda_1(x + y - 3) +$$
$$+ \lambda_2 x + \lambda_3 y$$

$$\underline{\underline{J}} = \begin{bmatrix} 1 & -2 \\ -1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{rank } \underline{\underline{J}} = 2 \qquad \checkmark$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 : \quad 3y - 3x^2 - \mu + \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 : \quad 3x + 2\mu + \lambda_1 + \lambda_3 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = 0 : \quad x - 2y + 2 = 0$$

$$\lambda_1(x + y - 3) = 0$$

$$\lambda_2 x = 0$$

$$\lambda_3 y = 0$$

$$x + y \geq 3$$

$$x \geq 0$$

$$y \geq 0$$

$$\lambda_1 \geq 0$$

$$\lambda_2 \geq 0$$

$$\lambda_3 \geq 0$$

1) $\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0$

2) $\lambda_1 = 0, \quad \lambda_2 = 0, \quad y = 0$

3) $\lambda_1 = 0, \quad x = 0, \quad \lambda_3 = 0$

4) $\lambda_1 = 0, \quad x = 0, \quad y = 0$

5) $x + y - 3 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0$

6) $x + y - 3 = 0, \quad \lambda_2 = 0, \quad y = 0$

7) $x + y - 3 = 0, \quad x = 0, \quad \lambda_3 = 0$

8) $x + y - 3 = 0, \quad x = 0, \quad y = 0$

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$$\begin{cases} 3y - 3x^2 - \mu = 0 \\ 3x + 2\mu = 0 \\ x - 2y + 2 = 0 \end{cases}$$

$$\begin{cases} \mu = 3y - 3x^2 \\ 3x + 2(3y - 3x^2) = 0 \\ x = 2y - 2 \end{cases}$$

$$3(2y-2) + 2[3y - 3(2y-2)^2] = 0$$

$$6y - 6 + 2[3y - 3(4y^2 - 8y + 4)] = 0$$

$$3y - 3 + [3y - 3 \cdot 4(y^2 - 2y + 1)] = 0$$

$$3y - 3 + 3[y - 4(y^2 - 2y + 1)] = 0$$

$$y - 1 + y - 4(y^2 - 2y + 1) = 0$$

$$2y - 1 - 4y^2 + 8y - 4 = 0$$

$$-4y^2 + 10y - 5 = 0$$

$$4y^2 - 10y + 5 = 0$$

$$y = \frac{5 \pm \sqrt{5}}{4}$$

$$x = 2\,\frac{5 \pm \sqrt{5}}{4} - 2 = \frac{5 \pm \sqrt{5}}{2} - 2 = \frac{1 \pm \sqrt{5}}{2}$$

$$\left(\frac{1+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$$

$$\left(\frac{1-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right)$$

$$\frac{1+\sqrt{5}}{2} \geqslant 0 \quad \checkmark$$

$$\frac{5+\sqrt{5}}{2} \geqslant 0 \quad \checkmark$$

$$x + y = \frac{1+\sqrt{5}}{2} + \frac{5+\sqrt{5}}{2} = \frac{6+2\sqrt{5}}{2} =$$

$$= 3 + \sqrt{5} \geqslant 3 \quad \checkmark$$

$A\left(\frac{1+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$ candidate point

$\frac{1-\sqrt{5}}{2} < 0$ so $\left(\frac{1-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right)$

is not acceptable.

(2)

$$\begin{cases} -3x^2 - \mu = 0 \\ 3x + 2\mu + \lambda_3 = 0 \\ x + 2 = 0 \end{cases}$$

$$\begin{cases} \mu = -3x^2 \\ -6 + 2\mu + \lambda_3 = 0 \\ x = -2 \end{cases}$$

$$\begin{cases} \mu = -12 \\ \lambda_3 = 6 - 2\mu \\ x = -2 \end{cases} \qquad \begin{cases} \mu = -12 \\ \lambda_3 = 30 \geq 0 \quad \checkmark \\ x = -2 \end{cases}$$

$(-2, 0)$

$x = -2 < 0$ so $(-2, 0)$ is not acceptable

(3)

$$\begin{cases} 3y - \mu + \lambda_2 = 0 \\ 2\mu = 0 \\ -2y + 2 = 0 \end{cases}$$

$$\begin{cases} 3y + \lambda_2 = 0 \\ \mu = 0 \\ 2y = 2 \end{cases} \qquad \begin{cases} \lambda_2 = -3 < 0 \\ \mu = 0 \\ y = 1 \end{cases}$$ not acceptable

(4)
$$\begin{cases} -\mu + \lambda_2 = 0 \\ 2\mu + \lambda_3 = 0 \\ 2 = 0 \end{cases} \quad \text{impossible}$$

Thus, also case (8) is impossible

(5)
$$\begin{cases} 3y - 3x^2 - \mu + \lambda_1 = 0 \\ 3x + 2\mu + \lambda_1 = 0 \\ x - 2y + 2 = 0 \\ x + y - 3 = 0 \end{cases}$$

$$\begin{cases} x - 2y + 2 = 0 \\ x + y - 3 = 0 \end{cases} \qquad \begin{cases} x = 2y - 2 \\ x = -y + 3 \end{cases}$$

$$\begin{cases} x = 2y - 2 \\ 2y - 2 = -y + 3 \end{cases} \qquad \begin{cases} x = 2y - 2 \\ 3y = 5 \end{cases} \qquad \begin{cases} x = \frac{4}{3} \geq 0 \quad \checkmark \\ y = \frac{5}{3} \geq 0 \quad \checkmark \end{cases}$$

$$\begin{cases} 3 \cdot \frac{5}{3} - 3 \cdot \frac{16}{9} - \mu + \lambda_1 = 0 \\ 3 \cdot \frac{4}{3} + 2\mu + \lambda_1 = 0 \end{cases}$$

$$\begin{cases} 5 - \frac{16}{3} - \mu + \lambda_1 = 0 \\ 4 + 2\mu + \lambda_1 = 0 \end{cases} \qquad \begin{cases} -\frac{1}{3} - \mu + \lambda_1 = 0 \\ 4 + 2\mu + \lambda_1 = 0 \end{cases}$$

$$\begin{cases} \lambda_1 = \mu + \frac{1}{3} \\ 4 + 2\mu + \mu + \frac{1}{3} = 0 \end{cases} \qquad \begin{cases} \lambda_1 = \mu + \frac{1}{3} \\ 3\mu + \frac{13}{3} = 0 \end{cases}$$

$$\begin{cases} \lambda_1 = -\frac{13}{9} + \frac{1}{3} = -\frac{13}{9} + \frac{3}{9} = -\frac{10}{9} < 0 \\ \mu = -\frac{13}{9} \end{cases} \qquad \text{not acceptable}$$

⑥ $\begin{cases} x + y - 3 = 0 \\ y = 0 \\ x - 2y + 2 = 0 \end{cases}$ $\begin{cases} x = 3 \\ y = 0 \\ x = -2 \end{cases}$ impossible

⑦ $\begin{cases} x + y - 3 = 0 \\ x = 0 \\ x - 2y + 2 = 0 \end{cases}$ $\begin{cases} y = 3 \\ x = 0 \\ y = 1 \end{cases}$ impossible

$A\left(\frac{1+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right)$ is the only candidate point

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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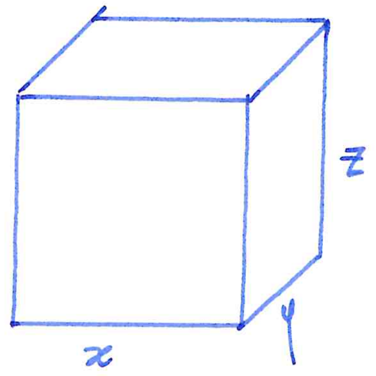
Among all the parallelepipeds with volume V, determine the one whose total surface is minimum. Is there a parallelepiped whose total surface is maximum?



$$S = 2(xz + xy + yz)$$

$$\min \quad 2xz + 2xy + 2yz$$
$$\text{s.t.:} \quad xyz = V$$

$$\max \quad -2xz - 2xy - 2yz$$
$$\text{s.t.:} \quad xyz = V$$

$$\mathcal{L} = -2xz - 2xy - 2yz - \mu(xyz - V)$$

$$h(x, y, z) = xyz$$

$$\nabla h = (yz, xz, xy) \neq (0, 0, 0)$$

$$(x, y, z) \notin \{(0,0,0), (a, 0, 0), (0, a, 0), (0, 0, a) \mid a \in \mathbb{R}\}$$

But this is never the case because if, for instance, $x = 0$ or $y = 0$ or $z = 0$, then the volume is 0 and not V.

$$\mathcal{L} = -2xz - 2xy - 2yz - \mu(xyz - V)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 : \quad -2z - 2y - \mu yz = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 : \quad -2x - 2z - \mu xz = 0$$

$$\frac{\partial \mathcal{L}}{\partial z} = 0 : \quad -2x - 2y - \mu xy = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = 0 : \quad -(xyz - V) = 0$$

$$\begin{cases} 2z + 2y + \mu yz = 0 \\ 2x + 2z + \mu xz = 0 \\ 2x + 2y + \mu xy = 0 \\ xyz = V \end{cases}$$

$$\begin{cases} \mu yz = -2(z+y) \\ \mu xz = -2(x+z) \\ \mu xy = -2(x+y) \\ xyz = V \end{cases}$$

$$\begin{cases} \mu y z = -2(z+y) \\ \dfrac{y}{x} = \dfrac{z+y}{x+z} \\ \dfrac{z}{y} = \dfrac{x+z}{x+y} \\ x y z = V \end{cases}$$

$$\begin{cases} \mu y z = -2(x+y) \\ y(x+z) = x(z+y) \\ z(x+y) = y(x+z) \\ x y z = V \end{cases}$$

$$\begin{cases} \mu y z = -2(x+y) \\ \cancel{yx} + yz = xz + \cancel{xy} \\ \cancel{zx} + zy = yx + \cancel{yz} \\ x y z = V \end{cases}$$

$$\begin{cases} \mu y z = -2(x+y) \\ y = x \\ z = x \\ x y z = V \end{cases}$$

$$\begin{cases} \mu = -\dfrac{\cancel{4\sqrt[3]{V}}}{\cancel{\sqrt[3]{V}}\,\sqrt[3]{V}} = -\dfrac{4}{\sqrt[3]{V}} \\ y = \sqrt[3]{V} \\ z = \sqrt[3]{V} \\ x = \sqrt[3]{V} \end{cases}$$

The set $\{(x, y, z) \in \mathbb{R}^3 : xyz = V, \ x > 0, \ y > 0, \ z > 0\}$ is neither closed nor compact.
It is not closed because we have constraints such as $x > 0$ (recall that we can't accept $x = 0$ otherwise the volume xyz becomes 0 instead of V).
It is neither bounded. In fact, we can have points in the set such as:

$$x = L, \quad y = L, \quad z = \frac{V}{L^2}$$

If $L \to +\infty$, then $z \to 0$ but both x and y approach $+\infty$
In this case, the total surface is:

$$S(x, y, z) = 2(xz + xy + yz)$$

$$S\left(L, L, \frac{V}{L^2}\right) = 2\left(\cancel{L} \cdot \frac{V}{L^{\cancel{2}}} + L^2 + L \cdot \frac{V}{L^{\cancel{2}}}\right) =$$

$$= 2\left(2\frac{V}{L} + L^2\right)$$

$$\lim_{L \to +\infty} S\left(L, L, \frac{V}{L^2}\right) = \lim_{L \to +\infty} 2\left(2\frac{V}{L} + L^2\right) = +\infty$$

Thus, there is not a parallelepiped whose total surface is maximum.