Mathematical methods for economics and finance

International Finance and Economics

Dept. of Economics and Law

Prof. Elisabetta Michetti

MOD B1 - Theory

FUNCTIONS OF SEVERAL REAL VARIABLES

1. Consider the law $y = e^x$

For all real values assigned to variable x, a unique real value of variable y is obtained. Hence $y = e^x$ is a function of one variable!

2. Consider the law $z = x^3 - y + 1$

For all real values assigned to variable x and for all real values assigned to variable y, a unique real value of variable z is obtained. To compute z we need to fix a value to variable x and a value to variable y, that is to fix the elements of the vector (x, y). Hence $z = x^3 - y + 1$ is a function of two real variables!

3. Consider the law $y = \sqrt{x}$

For all real values assigned to variable $x \geq 0$, a unique real value of variable $y \geq 0$ is obtained. Hence $y = \sqrt{x}$ is a function of one variable but in such a case x can assume only non-negative values, while the obtained values of variable y will not be negative!

4. Consider the law $z = \ln(xy)$

Again it is a function of two real variables, anyway the z-value can be determined if and only if (iff) the product $xy > 0$, that is x and y must be different from zero and they must have the same sign.

5. Consider the law $y = \frac{\sqrt{x_3(x_1x_2)^2}}{x_3}$

In this case to compute y we need to chose x_1, x_2 and x_3; furthermore we need to require that both $x_3(x_1x_2)^2 \geq 0$ and $x_3 \neq 0$ hold. Hence this is a function of three real variables and $(\underline{x}) = (x_1, x_2, x_3)$ must be taken in the following set: $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$, representing the set of vectors in $\mathbb{R}^3$ having the third component positive.

6. Consider the law $y = e^{x_1 + x_2^3} + |x_3 - \ln(x_4^2 + 1)|$

It is a function of four real variables: the y value, which depends on $(\underline{x})$, can be computed for all $(\underline{x}) \in \mathbb{R}^4$ but in all cases a non negative number will be obtained!

Those are examples of functions of one or more real variables!

Def. FUNCTION

A FUNCTION $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is a rule (or law) that assigns to each vector in a set A, one and only one number in $\mathbb{R}$. The corresponding rule can be denoted by $y = f(\underline{x})$.

- $A \subseteq \mathbb{R}^n$ is the DOMAIN,
- $\mathbb{R}$ is the CODOMAIN (or target set),
- $\underline{x} = (x_1, x_2, ..., x_n)$ is the INDEPENDENT VARIABLE,
- $y \in \mathbb{R}$ is the DEPENDENT VARIABLE,
- $Im_f = \{y \in \mathbb{R} : y = f(\underline{x}) \forall \underline{x} \in A\}$ is the IMAGE SET.

Coming back to the previous examples...

1. Consider the law $y = e^x$

The domain is $A = \mathbb{R}$, the codomain is $\mathbb{R}$ while the image set is $\mathbb{R}_+ - \{0\} = (0, +\infty)$.

2. Consider the law $z = x^3 - y + 1$

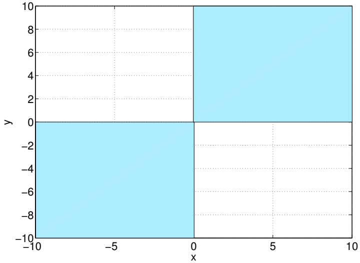
The domain is $A = \mathbb{R}^2$, the codomain is $\mathbb{R}$ and also the image set is $\mathbb{R}$.

3. Consider the law $y = \sqrt{x}$

As the square root of a negative number cannot be computed, the domain is $A = \mathbb{R}_+ = [0, +\infty)$, the codomain is $\mathbb{R}$ while the image set is $Im_f = \mathbb{R}_+$.

4. Consider the law $z = \ln(xy)$

Since only the logarithm of a positive number can be calculated, then the domain is $A = \{(x, y) \in \mathbb{R}^2 : xy > 0\}$, the codomain and the image set are both $\mathbb{R}$. In such a case the domain can be colored on the plane (x, y) as in the figure.



Homeworks

Determine domain, codomain and image sets of the following functions.

- $y = \sqrt{(x+2)}$,
- $z = \ln(y - x^2)$ and $z = \sqrt{(y - x)}$.
- $y = e^{x_1} \ln(x_1(x_2 + x_3 + 1)^2)$.

EXAMPLES OF FUNCTIONS OF SEVERAL REAL VARIABLES IN ECONOMICS AND FINANCE...

The demand function

$$q_1 = f(p_1, p_2, y)$$

The quantity demanded by a consumer of good 1 given by q_1 depends on the price of good 1 namely $p_1 \geq 0$, the price of good 2 namely $p_2 \geq 0$ and the disposable income given by $y \geq 0$.
Hence $f : A \subseteq \mathbb{R}^3_+ \to \mathbb{R}$.

...EXAMPLES OF FUNCTIONS OF SEVERAL REAL VARIABLES IN ECONOMICS AND FINANCE

The utility function

$$u = f(x_1, x_2, ..., x_n)$$

The quantity consumed of good 1, 2, ..., n are given by $x_1 \geq 0$, $x_2 \geq 0$,...,$x_n \geq 0$ while u is the utility assigned by a consumer to the pannier.
Hence $f : A \subseteq \mathbb{R}^n_+ \to \mathbb{R}$.

...EXAMPLES OF FUNCTIONS OF SEVERAL REAL VARIABLES IN ECONOMICS AND FINANCE

The production function

$$y = f(x_1, x_2, ..., x_n)$$

Here x_1, x_2, ..., x_n are the not neative quantities of inputs used in the production process (such as capital, labour, infrastructures, technology ...) and y is the correspondent production level.
Hence $f : A \subseteq \mathbb{R}^n_+ \to \mathbb{R}$.

...EXAMPLES OF FUNCTIONS OF SEVERAL REAL VARIABLES IN ECONOMICS AND FINANCE

Expected return of a portfolio

$$R_e = f(x_1, x_2, ..., x_n, R_1, R_2, ...R_n)$$

Here x_1, x_2, ..., x_n are the fractions of assets $1, 2, ..., n$ in the portfolio, that is $x_i \in [0, 1], \forall i = 1, 2, ..., n$ and $x_1 + x_2 + ... + x_= 1$, while R_1, R_2,...,R_n are the expected return of each asset (it is normally given, hence they are constants).
Hence $f : A \subseteq \mathbb{R}^n_+ \to \mathbb{R}$.

EXAMPLES OF UTILITY OR PRODUCTION FUNCTIONS tipically used in applications

- Linear function: $y = a_1x_1 + a_2x_2 + ... + a_nx_n$
- Cobb-Douglas function: $y = Ax_1^{b_1}x_2^{b_2}...x_n^{b_n}$
- Leontief function: $y = \min\left\{\frac{x_1}{c_1}, \frac{x_2}{c_2}, ..., \frac{x_n}{c_n}\right\}$
- CES (constant elasticity of substitution) function of two factors: $y = K(c_1x_1^{-a} + c_2x_2^{-a})^{\frac{-b}{a}}$,

where $\underline{x} \in A \subseteq \mathbb{R}_+^n$: $\underline{x}$ is the independent variable and A is the domain,
$y \geq 0$, y is the dependent variable, and the image is a subset of $\mathbb{R}_+$.
All the constants are positive.

Def. GRAPH

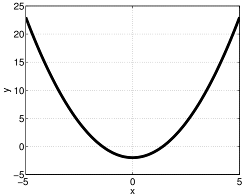
Let $y = f(\underline{x}) = f(x_1, x_2, ..., x_n)$ be a function of n real variables defined on the domain A. The GRAPH of function f is given by

$$G_f = \{(x_1, x_2, ..., x_n, y) \in \mathbb{R}^{n+1} : y = f(\underline{x}), \forall \underline{x} \in A\}.$$

If $n = 1$, that is $y = f(x)$, then $G_f \in \mathbb{R}^2$. Its graph is a curve on the plane (x, y) and it can be qualitatively determined and depicted.

$y = x^2 - 2$

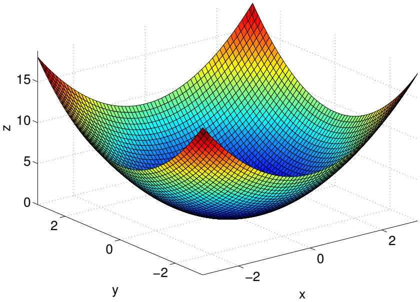
The domain is $\mathbb{R}$ and from the elementary function $y = x^2$ its graph can be easily obtained.



If $n = 2$, that is $z = f(x, y)$, then $G_f \in \mathbb{R}^3$. Its graph is a surface of the 3−dimensional space and it can be difficult to be drawn.

$z = x^2 + y^2$

Its domain is $\mathbb{R}^2$ and its graph is the following.



If $n > 2$, then $G_f \in \mathbb{R}^{n+1}$ and its graph cannot be drawn!

In order to know the graph of functions of two variables it is of great help to define the level curves.

Def. LEVEL CURVES

Let $z = f(x, y)$ and consider $z = z_0 \in Im_f$. Then the locus $(x, y) \in A$ such that $f(x, y) = z_0$ is said level curve. While moving the fixed value of z_0 several curves can be drawn, called LEVEL CURVES.

$z = x^2 + y^2$

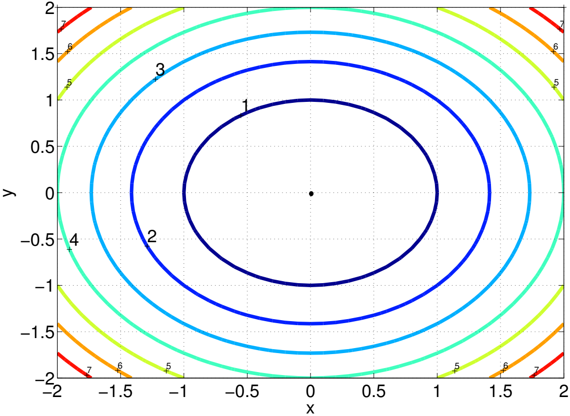
Then $A = \mathbb{R}^2$ while $Im_f = \mathbb{R}_+$.

If $z_0 = 0$ then $x^2 + y^2 = 0$ iff $x = 0$ and $y = 0$ hence the level curve is the origin!

If $z_0 = 1$ then $x^2 + y^2 = 1$ is a circumference with centre in $(0, 0)$ and radius 1.

For all $z_0 > 0$ then $x^2 + y^2 = z_0$ describes a circumference with centre in $C = (0, 0)$ and radius $r = \sqrt{z_0}$.
Thus the radius increases as z_0 increases.

The level curves of $z = x^2 + y^2$ are the following.



Examples

We want to depict the level curves of the following functions of two real variables.

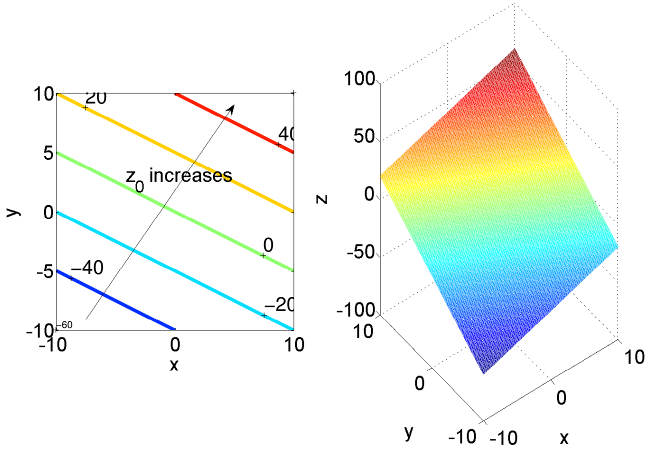
(a) $z = 2x + 4y$;

(b) $z = y - \ln(x)$.

(a) $z = 2x + 4y$

Then $A = \mathbb{R}^2$ and $Im_f = \mathbb{R}$.
For all z_0, from $z_0 = 2x + 4y$ one obtains $y = \frac{z_0}{4} - \frac{x}{2}$ that are straight lines.
The corresponding graph of the function is a plane.

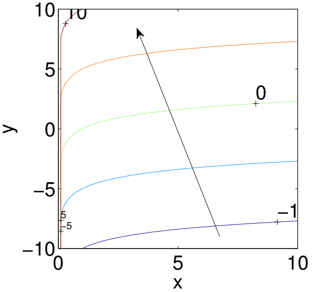
The level curves of $z = 2x + 4y$ and its graph are the following.

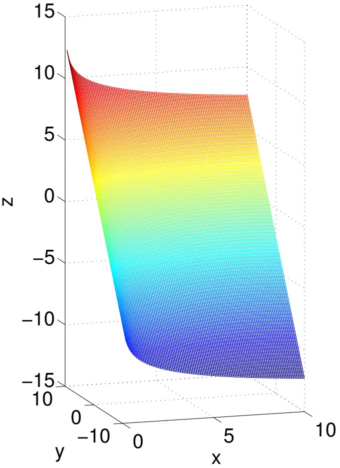


(b) $z = y - \ln(x)$

Then $A = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $Im_f = \mathbb{R}$.
For all z_0, from $z_0 = y - \ln(x)$ one obtains $y = z_0 + \ln(x)$ that can be easily depicted by traslations of function $y = \ln(x)$.

The level curves of $z = y - \ln(x)$ and its graph are the following.





INDIFFERENCE CURVES

If $z = f(x, y)$ is a utility function then the level curves are said INDIFFERENCE CURVES. They represent the combinations of consumed quantities of each good corresponding to the same utility.

ISOQUANTS

If $z = f(x, y)$ is a production function then the level curves are said ISOQUANTS. They represent the combinations of quantities of each production factor corresponding to the same production level.

Example

We want to depict the indifference curves of the following Cobb-Douglas utility function: $y = \sqrt{x_1 x_2}$.

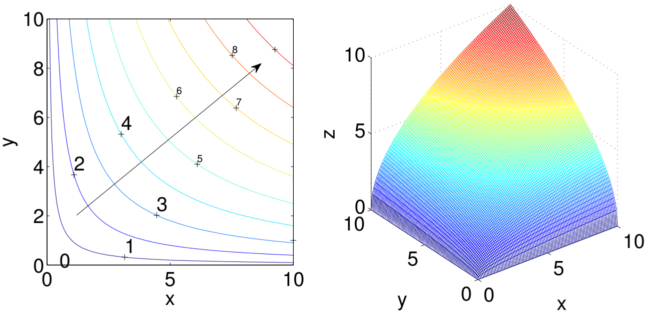
$y = \sqrt{x_1 x_2}$

Then $A = \mathbb{R}^2_+$ and $Im_f = \mathbb{R}_+$.
For all $y_0 \geq 0$, form $y_0 = \sqrt{x_1 x_2}$ one gets $(y_0)^2 = x_1 x_2$.

If $y_0 = 0$ then the two semiaxis are obtained.

If $y_0 > 0$ then it can be obtained function $x_2 = \frac{(y_0)^2}{x_1}$ that can be easily depicted by traslating the elementary function $y = 1/x$ (iperbolae).

The level curves of $y = \sqrt{x_1 x_2}$ and its graph are the following.



Homeworks

- Level curves of functions $z = y - e^x$ and $z = -y + x^3 + 1$.
- Indifference curves of Cobb-Douglas function $z = x^2 y$.
- Isoquant of linear function $z = 4x + y$.

Notice that if function f has a complicated analytical form, then the level curves cannot be easily depicted and the graph cannot be easily reached.

To the scope we will introduce software **MatLab.**