International Finance and Economics
Dept. of Economics and Law

Mathematical methods for economics and finance

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PART 3 - Theory

Def: Distance between two vectors

Let $\underline{x} = (x_1, \ldots, x_n) \in \mathrm{R}^n$, $\underline{y} = (y_1, \ldots, y_n) \in \mathrm{R}^n$.

The DISTANCE between $\underline{x}$ and $\underline{y}$ is the following not negative number:

$$d(\underline{x}, \underline{y}) = \left\| \underline{x} - \underline{y} \right\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Def: Neighborhood of $\underline{x}_0$

Let $\underline{x} \in R^n$ and $r \in R, r > 0$

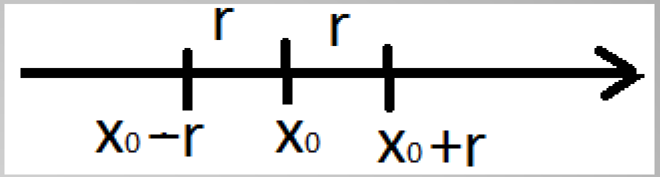
A NEIGHBORHOOD of $\underline{x}_0$ with radius r is given by:

$$B(\underline{x}_0, r) = \left\{ \underline{x} \in R^n : d(\underline{x}_0, \underline{x}) < r \right\}$$

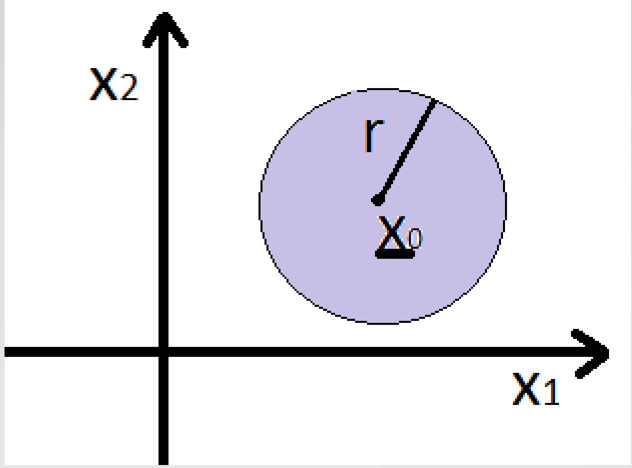
Notice: We call the set $B(\underline{x}_0, r)$, r-BALL about $\underline{x}_0$

EX: x_0 belongs to R

$$\Rightarrow \mathrm{B}(x_0, r) = (x_0 - r, x_0 + r)$$



EX: $\underline{x}_0$ belongs to R^2



Unconstrained optimization

Def: absolute (or global) maximum point and absolute (or global) minimum point

Let $f : A \subseteq R^n \to R$ and $\underline{x}^* \in A$

$\underline{x}^*$ is an ABSOLUTE MAXIMUM (MAX) point if

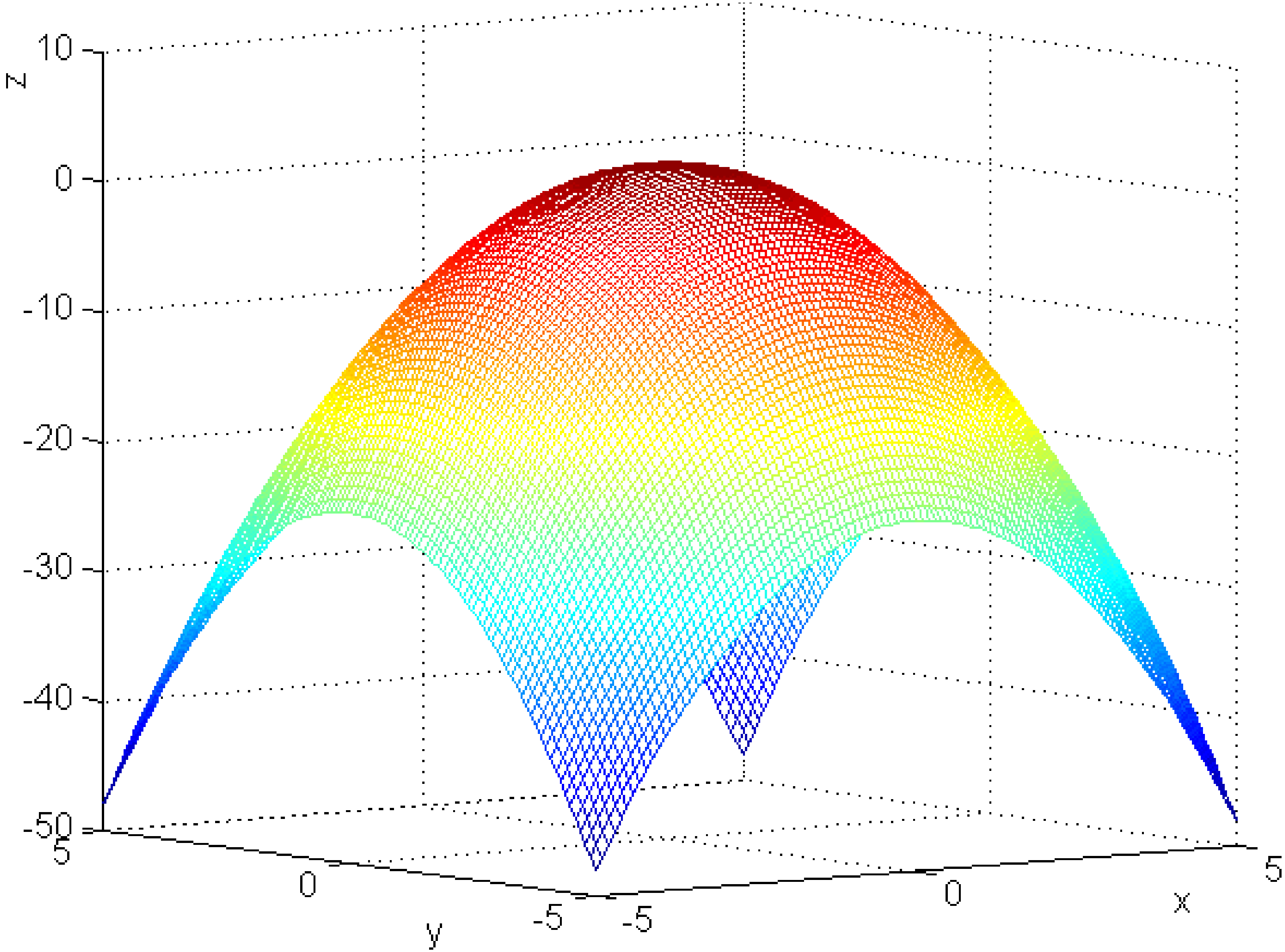
$$f(\underline{x}^*) \geq f(\underline{x}) \qquad \forall \underline{x} \in A$$

$\underline{x}^*$ is an ABSOLUTE MINIMUM (MIN) point if

$$f(\underline{x}^*) \leq f(\underline{x}) \qquad \forall \underline{x} \in A$$

Notice: If a point is an absolute max then there are no points in the domain at which f takes a larger value

EX: absolute maximum



Def: relative (or local) maximum point and relative (or local) minimum point

Let $f : A \subseteq R^n \to R$ and $\underline{x}^* \in A$

$\underline{x}^*$ is a RELATIVE MAXIMUM point if

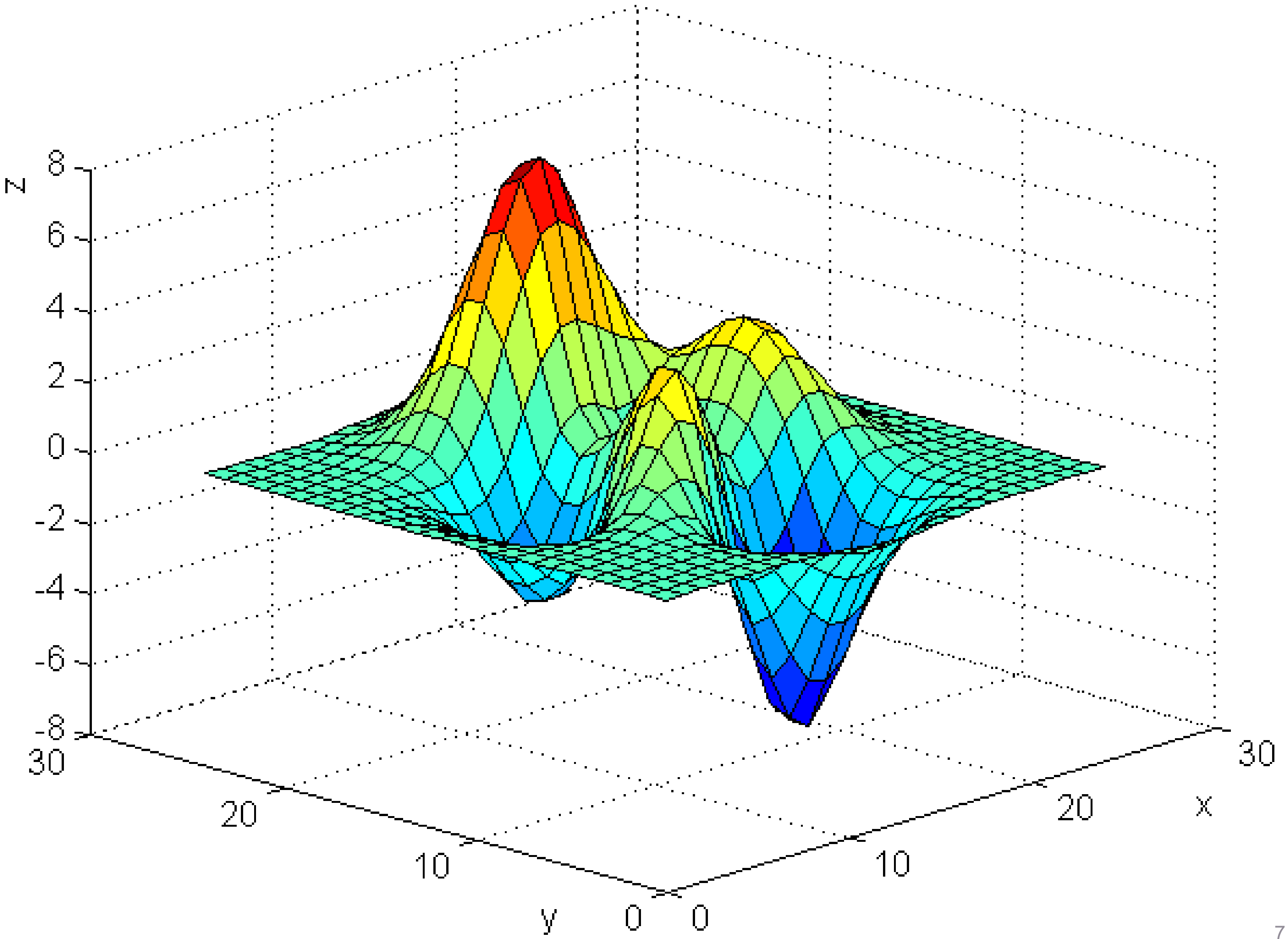
$$\exists\, B(\underline{x}^*, r) : \; f(\underline{x}^*) \geq f(\underline{x}) \quad \forall \underline{x} \in B(\underline{x}^*, r) \cap A$$

$\underline{x}^*$ is an RELATIVE MINIMUM point if

$$\exists\, B(\underline{x}^*, r) : \; f(\underline{x}^*) \leq f(\underline{x}) \quad \forall \underline{x} \in B(\underline{x}^*, r) \cap A$$

Notice: If a point is a local max then there are no nearby points at which f takes a larger value

EX: local maximum and minimum



The **main goal** of this section it to give an answer to the following problem.

Let y=f(x) be a function of several variables, we want to determine its local maximum and local minimum points.

Notice: **we will give an answer to this problem while considering functions f having some properties that are usually verified in economics.**

Preliminarly we give some definitions extending those given for functions of one real variable.

Def: LIMIT of a function

1)A point $\underline{x} \in R^n$ is an **accumulation point** of $A \subseteq R^n$
if in all r-balls of $\underline{x}$ there exists a point of A different from $\underline{x}$

2) Let $f : A \subseteq R^n \to R$ and let $\underline{x}^* = (x_1^*, \ldots, x_n^*) \in A$
be an accumulation point of A. Then

$$\lim_{\underline{x} \to \underline{x}^*} f(x_1, \ldots, x_n) = l$$

if $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $\| f(\underline{x}) - l \| < \varepsilon$

$$\forall \underline{x} \in B(\underline{x}^*, \delta) \cap A - \{\underline{x}^*\}$$

Def: CONTINUITY of a function

Let $f : A \subseteq R^n \to R$ and $\underline{x}^* \in A$ an accumulation point of A.

f is **continuous in $\underline{x}^*$** if

$$\lim_{\underline{x} \to \underline{x}^*} f(x_1, \ldots, x_n) = f(\underline{x}^*) = f(x_1^*, \ldots, x_n^*)$$

Notice: Function *f* is continuous in set A if it is continuous in all points of set A

We will consider only continuous functions!

Def: PARTIAL DERIVATIVE

Let $f : A \subseteq R^n \to R$ and $\underline{x}^* \in A$.

The PARTIAL DERIVATIVE of f with respect to variable x_i is given by the following limit as long as it EXISTS and it is FINITE

$$\lim_{x_i \to x_i^*} \frac{f(x_1^*,\ldots,x_{i-1}^*,x_i,x_{i+1}^*\ldots,x_n^*) - f(x_1^*,\ldots,x_{i-1}^*,x_i^*,x_{i+1}^*\ldots,x_n^*)}{x_i - x_i^*}$$

We can write it as: $f_{x_i}(\underline{x}^*)$ or $\dfrac{\partial f}{\partial x_i}(\underline{x}^*)$

Def: GRADIENT VECTOR

If function f $f : A \subseteq R^n \to R$

admits n partial derivatives in a point $\underline{x}^* \in A$,

the vector containing the derivatives of f in that point

is called GRADIENT VECTOR and it is indicated by ∇f

$$\nabla f(\underline{x}^*) = \left(\frac{\partial f}{\partial x_1}(\underline{x}^*), \frac{\partial f}{\partial x_2}(\underline{x}^*), \ldots, \frac{\partial f}{\partial x_n}(\underline{x}^*) \right)$$

Def: function of CLASS C^1

If all the partial derivatives of function $f : A \subseteq R^n \rightarrow R$ are continuous in a point $\underline{x}^* \in A$

$$\Downarrow$$

f is said the be ***of class*** C^1 in $\underline{x}^*$

We will consider only C^1 functions!

Ex1: consider the following functions

$$(1)\, y = x_1^2 + 2x_2^2 + 5x_1x_2, \quad (2)\, y = 2\sqrt{x_1} + 3x_1x_3 + e^{x_2}$$

The gradient vectors are given by:

$$(1)\, \nabla y = (2x_1 + 5x_2, 4x_2 + 5x_1),$$

$$(2)\, \nabla y = (1/\sqrt{x_1} + 3x_3, e^{x_2}, 3x_1).$$

The gradient of (1) in point (1,2) is

$$(1)\, \nabla y(1,2) = (12,13)$$

While the gradient of (2) in point (1,0,2) is given by

$$(2)\, \nabla y(1,0,2) = (7,1,3).$$

Homeworks

EX 1.1

(1) Consider the following function

$$y = x_1^2 x_3 - x_2^3 x_1 + 5x_2$$

and determine the gradient vector in points (1,2,1) and (0,3,-1).

(2) Consider the following function

$$z = e^{x^2 y} - \ln(x+1) + \sqrt{y}$$

and determine the gradient vector.

Homeworks

EX 1.2

Determine the domain and the gradient vector of the following functions:

$$(1)\, y = \sqrt{x^2 - 1}$$

$$(2)\, z = x^3 y^4 + 3xy^2 - 2y$$

$$(3)\, z = \ln(x^2 - y) + \sqrt{3x}$$

$$(4)\, y = e^{3x_1 - 2} + 4x_1 x_3^2 - \frac{1}{x_2}$$

Def. INTERIOR POINT

A point $\underline{x}^*$ is an interior point of A if there exists a whole r-ball about $\underline{x}^*$ in the domain A.

First order condition: THEOREM

Let $f : A \subseteq R^n \rightarrow R$ **be a C^1** function and $\underline{x}^* \in$ A is an interior point of A.

If $\underline{x}^*$ is a local max or min of f then

$$\frac{\partial f}{\partial x_i}(\underline{x}^*) = 0 \quad i = 1, \dots, n$$

Def: Critical point

An interior point $\underline{x}$ is said to be a **critical point** if for all i

$$\frac{\partial f}{\partial x_i}(\underline{x}) = 0$$

The previous theorem states a **necessary condition** for an interior point being a relative maximum or minimum point.

The points that can be local max or min must be investigated between **points belonging to the boundary of the domain A** or the **critical points**.

Anyway the previous condition **is not sufficient** since if $\underline{x}^*$ is a critical points then it is not necessarely a local max or min.

Ex2: determine the critical points of function

$$z = x^3 - 3x + y^2 - 4y$$

The domain A is R^2 and all points in A are interior points.

The partial derivatives are:

$$z_x = 3x^2 - 3, z_y = 2y - 4$$

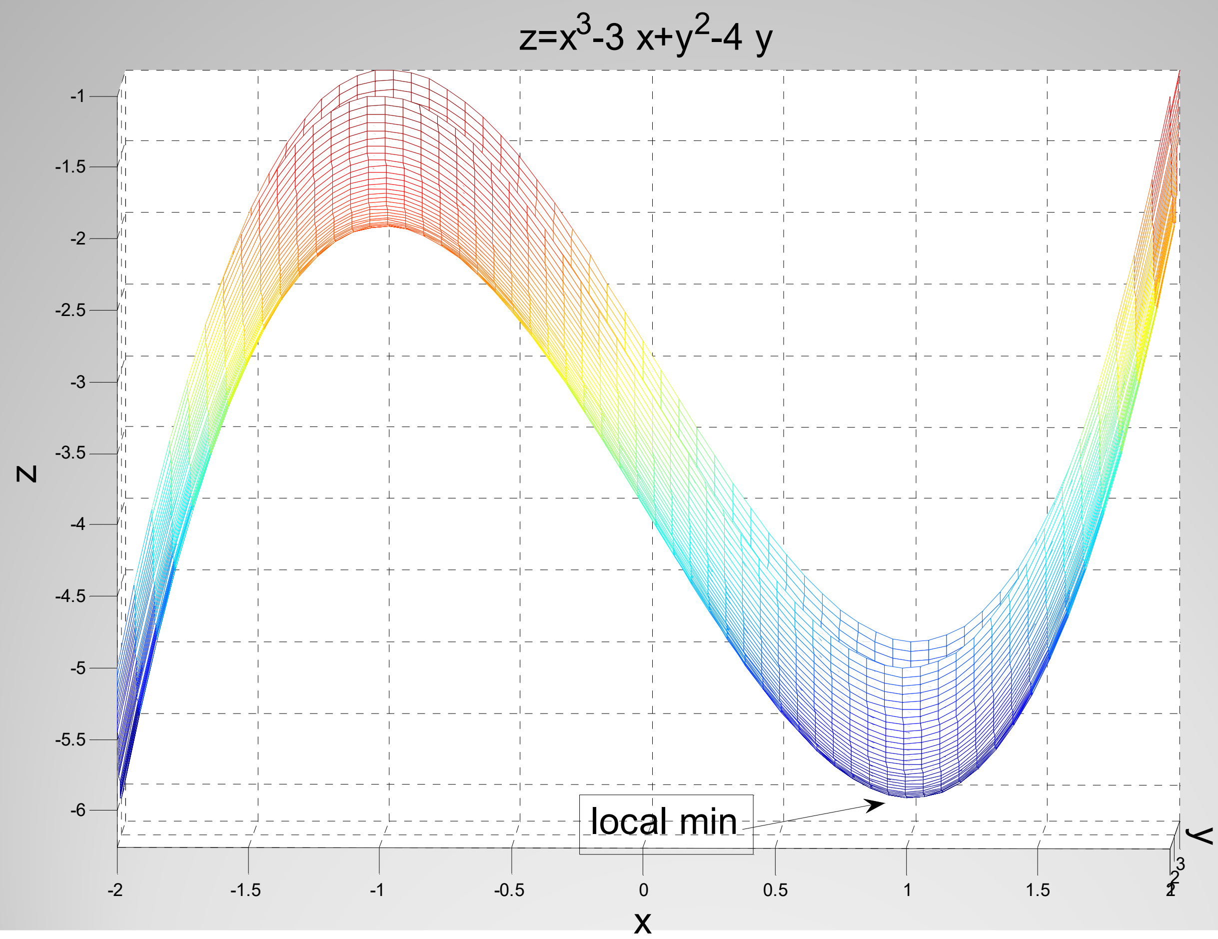
The critical points can be found by solving the following system:

$$\begin{cases} 3x^2 - 3 = 0 \\ 2y - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = \pm 1 \\ y = 2 \end{cases}$$

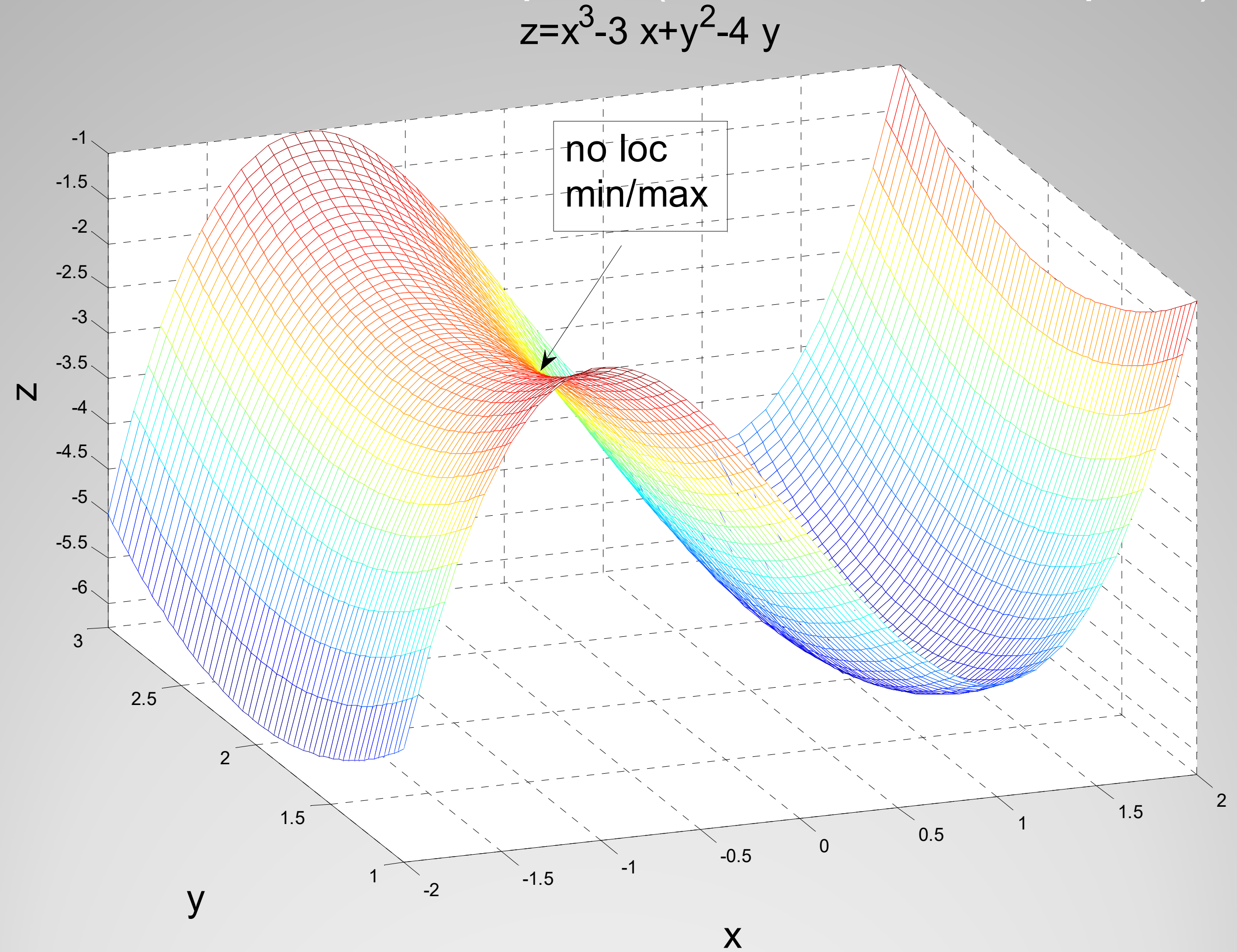
Thus points P=(1,2) and Q=(-1,2) are critical points of f.

They **can be** local max or min points.

From the graph of f it can be observed that the critical point P is a local minimum point



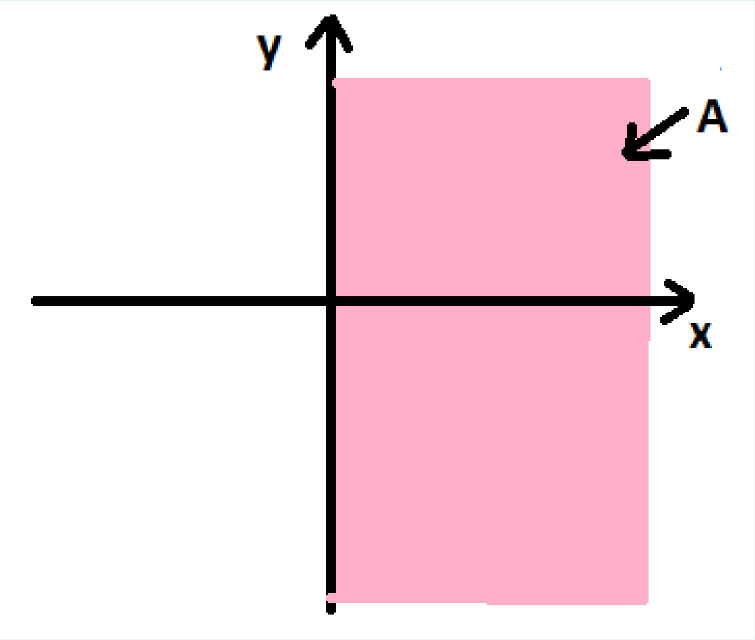
From the graph of f it can be observed that the critical point Q is not a local minimum/maximum point (it is called saddle point)



Ex3: determine the critical points of function

$$z = 2\sqrt{x} - x + 2y^4$$

The domain A is the set of points having x≥0 (that is the semi-plane with not-negative x values) as below.



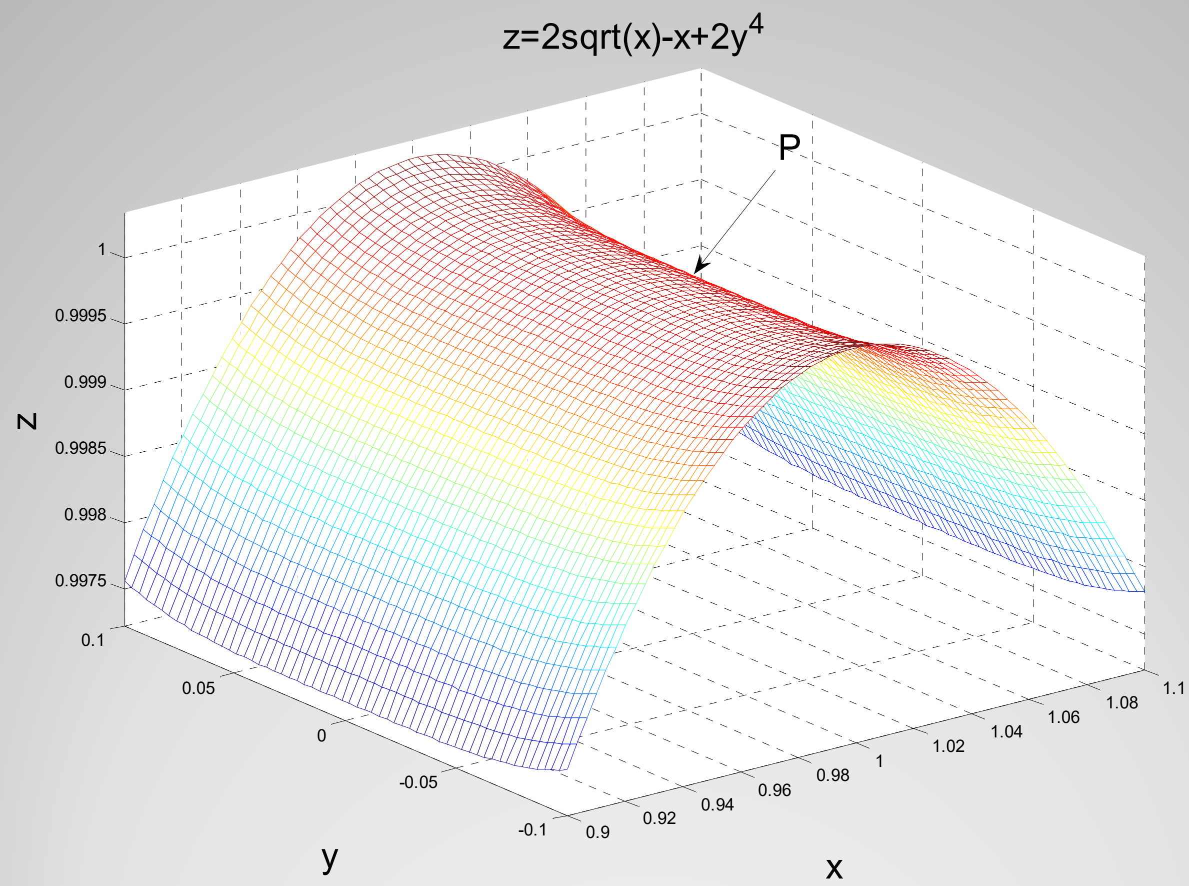
The partial derivatives are: $z_x = \frac{1}{\sqrt{x}} - 1, z_y = 8y^3$

The critical points can be found by solving the following system:

$$\begin{cases} \frac{1-\sqrt{x}}{\sqrt{x}} = 0 \\ 8y^3 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \end{cases} \Rightarrow P = (1,0)$$

P **can be** local max or min point. But **in addition local max or min points can belong to the border of A**, that is the set x=0.

From the graph of f it can be observed that the critical point P is not a local minimum/maximum (it is a saddle point)



Ex4: determine the critical points of function

$$z = \ln(x_1) - x_1 + x_2 x_3^2 - x_2^2 - x_2$$

The domain A is the set of points having $x_1>0$, that is $A = \{(x_1, x_2, x_3) \in R^3 : x_1 > 0\}$. Hence all points in A are interior points, while points that do not belong to A cannot be considered. To determine the critical points, the following system must be solved.

$$\begin{cases} \dfrac{\partial y}{\partial x_1} = \dfrac{1}{x_1} - 1 = \dfrac{1 - x_1}{x_1} = 0 \\ \dfrac{\partial y}{\partial x_2} = x_3^2 - 2x_2 - 1 = 0 \\ \dfrac{\partial y}{\partial x_3} = 2x_2 x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_3^2 - 2x_2 - 1 = 0 \\ x_2 = 0 \text{ or } x_3 = 0 \end{cases}$$

Two different systems must be consider in order to find **all the solutions**:

$$I\begin{cases} x_1 = 1 \\ x_3^2 - 2x_2 - 1 = 0 \\ x_2 = 0 \end{cases} \Rightarrow I\begin{cases} x_1 = 1 \\ x_3^2 - 1 = 0 \\ x_2 = 0 \end{cases} \Rightarrow I\begin{cases} x_1 = 1 \\ x_3 = \pm 1, \\ x_2 = 0 \end{cases}$$

$$II\begin{cases} x_1 = 1 \\ x_3^2 - 2x_2 - 1 = 0 \\ x_3 = 0 \end{cases} \Rightarrow I\begin{cases} x_1 = 1 \\ -2x_2 - 1 = 0 \\ x_3 = 0 \end{cases} \Rightarrow II\begin{cases} x_1 = 1 \\ x_2 = -\dfrac{1}{2} \\ x_3 = 0 \end{cases}$$

And three solutions are found all belonging to A. The critical points are: M=(1,0,1), N=(1,0,-1) and P=(1,-1/2,0)

Homeworks

EX 1.3

Determine the critical points of the following functions:

$$(1) z = 3x^2 - 2y^2 + 6xy - 12x$$

$$(2) z = xe^y - x - y$$

$$(3) y = x_1^2(x_2 - 1) + x_2^2 x_3 - 4x_2$$

$$(4) y = (x_3 - 2)^2 + x_1(x_2 - 3)$$

$$(5) z = \ln x - 2x^2 - 4(y-5)^2$$

$$(6) y = 3x_1 + 5x_2^4 x_3 + x_3 x_4$$

THE SECOND PARTIAL DERIVATIVES

If all the first partial derivatives are derivable again, then it is possible to calculate their partial derivatives thus obtaining:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i x_j} \quad \text{with } i \neq j$$

Mixed second derivative

$$\frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_i^2} = f_{x_i x_i}$$

Pure second derivative

Def: function of CLASS C^2

If all the second derivatives of f exist and are continuous, then f is said to be of C^2 class

We will consider only C^2 functions!

Schwarz THEOREM

If $f : A \subseteq R^n \to R,\ A$ open set, is a C^2 function on A then

$$\Downarrow$$

$\forall \underline{x} \in A$ and $\forall\ i, j$

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\underline{x}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x})$$

Def: Hessian of f in point $\underline{x}^*$

Let f be of C^2 class and let $\underline{x}^*$ be an interior fixed point. The hessian of f in point $\underline{x}^*$ is given by:

$$Hf(\underline{x}^*) = \begin{pmatrix} f_{x_1x_1}(\underline{x}^*) & f_{x_1x_2}(\underline{x}^*) & \cdots & f_{x_1x_n}(\underline{x}^*) \\ f_{x_2x_1}(\underline{x}^*) & f_{x_2x_2}(\underline{x}^*) & \cdots & f_{x_2x_n}(\underline{x}^*) \\ \vdots & \vdots & \cdots & \vdots \\ f_{x_nx_1}(\underline{x}^*) & f_{x_nx_2}(\underline{x}^*) & \cdots & f_{x_nx_n}(\underline{x}^*) \end{pmatrix}$$

Notice that Hf is a symmetric square matrix (nxn).

Ex5: consider the following function $y = 3x_1^3 - x_2^2 x_3$

The first partial derivatives are given by:

$$y_{x_1} = 9x_1^2, y_{x_2} = -2x_2x_3, y_{x_3} = -x_2^2$$

The second partial derivatives are given by:

$$y_{x_1x_1} = 18x_1, \quad y_{x_1x_2} = 0 = y_{x_2x_1}, \quad y_{x_1x_3} = 0 = y_{x_3x_1},$$

$$y_{x_2x_1} = 0 = y_{x_1x_2}, \quad y_{x_2x_2} = -2x_3, \quad y_{x_2x_3} = -2x_2 = y_{x_3x_2},$$

$$y_{x_3x_1} = 0 = y_{x_3x_1}, \quad y_{x_3x_2} = -2x_2 = y_{x_3x_2}, \quad y_{x_3x_3} = 0$$

Hence the Hessian matrix is:

$$Hf(\underline{x}) = \begin{pmatrix} 18x_1 & 0 & 0 \\ 0 & -2x_3 & -2x_2 \\ 0 & -2x_2 & 0 \end{pmatrix}$$

While the Hessian matrix in point (1,2,3) is:

$$Hf(\underline{x}) = \begin{pmatrix} 18 & 0 & 0 \\ 0 & -6 & -4 \\ 0 & -4 & 0 \end{pmatrix}$$

Homeworks

EX 1.4

1. Consider the following function $y = x_1 x_2^4 + x_3^4 x_4^3 - x_3 x_2$ and determine the Hessian matrix

2. Consider the following function $y = 2x_1^2 x_3 + x_2^3 x_1 - 5x_2 x_3$ and determine the Hessian matrix in point (1,0,-2)

Def: Definition of a symmetrix matrix

Let $A=[a_{ij}]$ be a symmetric matrix (nxn). We recall that it admits only real eigenvalues and the following definition holds.

A is:

Positive definite iff all the eigenvalues of A are positive,

Negative definite if all the eigenvalues of A are negative,

Positive semidefinite if all the eigenvalues of A are not negative and at least one is zero

Negative semidefinite if all the eigenvalues of A are not positive and at least one is zero

Indefinite if A admits both positive and negative eigenvalues

Ex6: The following matrix B is indefinite, in fact:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$|B - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & -1 & -\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-\lambda) - (1-\lambda) =$$

$$(1-\lambda)(\lambda^2 - 2\lambda - 1) = 0 \Leftrightarrow \lambda = 1 \text{ or } (\lambda^2 - 2\lambda - 1) = 0$$

that is $\lambda = 1$ or $\lambda = \dfrac{2 \pm \sqrt{8}}{2} \Rightarrow$

Two positive eigenvalues and one negative eigenvalue

Ex7: The following matrix C is indefinite, in fact with MatLab:

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

```
>> C=[-1 0 0; 0 2 -1; 0 -1 0]
C =
   -1     0     0
    0     2    -1
    0    -1     0
>> eig(C)
ans =
  -1.0000
  -0.4142
   2.4142
```

Homeworks

EX 1.5

Determine the definition of the following matrices:

$$B = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & 0 \end{pmatrix} \text{analitically and with MatLab}$$

$$C = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \text{analitically and with MatLab}$$

$$D = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \text{with MatLab}$$

Def: Definition SADDLE POINT

An interior point $\underline{x}^* \in A$ is a SADDLE POINT of $f : A \subseteq R^n \to R$, if $\forall B(\underline{x}^*, r)$,

there exists points $\underline{x} \in B(\underline{x}^*, r) \cap A$ **such that** $f(\underline{x}) > f(\underline{x}^*)$

and there exists points $\underline{x} \in B(\underline{x}^*, r) \cap A$ **such that** $f(\underline{x}) < f(\underline{x}^*)$

Second order condition: THEOREM

Let $f : A \subseteq R^n \to R$ **be a** C^2 function and $\underline{x}^* \in A$ is an interior cirtical point of A.

(1) If the Hessian Hf($\underline{x}^*$) is a negative definite matrix then $\underline{x}^*$ is a relative MAX of f

(2) If the Hessian Hf($\underline{x}^*$) is a positive definite matrix then $\underline{x}^*$ is a relative MIN of f

(3) If the Hessian Hf($\underline{x}^*$) is indefinite then $\underline{x}^*$ is neither a relative MAX nor a relative MIN of f. It is a SADDLE POINT.

Notice that: the previous Theorem states only a sufficient condition!

In fact, if the Hessian matrix is **semi-definite** in an interior critical point, then nothing can be said about the nature of that critical point!

We will solve analytically some problems of Unconstrained Optimization that are not TOO COMPLEX.

Ex8: Determine the local max and min points of the following function: $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2x_1 - 2x_3 - 5$

The **critical points** are given by the solutions of the following system:

$$\begin{cases} f_{x_1} = 2x_1 - 2 = 0 \\ f_{x_2} = 2x_2 = 0 \\ f_{x_3} = 2x_3 - 2 = 0 \end{cases} \Rightarrow P = (1,0,1)$$

The **Hessian matrix** in point P is given by:

$$Hf(x_1, x_2, x_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = Hf(1,0,1)$$

$Hf(1,0,1)$ is a diagonal matrix
hence its eigenvalues are given by the
elements belonging to the main diagonal
that **are all positive**

HENCE

(1,0,1) is a LOCAL MINIMUM point of f

Ex9: Determine the local max and min points of the following function: $z = \ln x - 2x^2 + y^4 - 32y$

The domain is given by the points (x,y) having x>0. All points in the domain are interior points. The **critical points** are given by the feasible solutions of the following system:

$$\begin{cases} z_x = \dfrac{1}{x} - 4x = 0 \Rightarrow \dfrac{1-4x^2}{x} = 0 \Rightarrow 4x^2 = 1 \Rightarrow x = \pm\dfrac{1}{2} \\ z_y = 4y^3 - 32 = 0 \Rightarrow 4y^3 = 32 \Rightarrow y^3 = 8 \Rightarrow y = 2 \end{cases}$$

Only the point (1/2,2) is a critical point since (-1/2,2) cannot be considered. In fact (-1/2,2) does not belong to the domain so that **(-1/2,2) is an unfeasible point**!

The Hessian Matrix is given by:

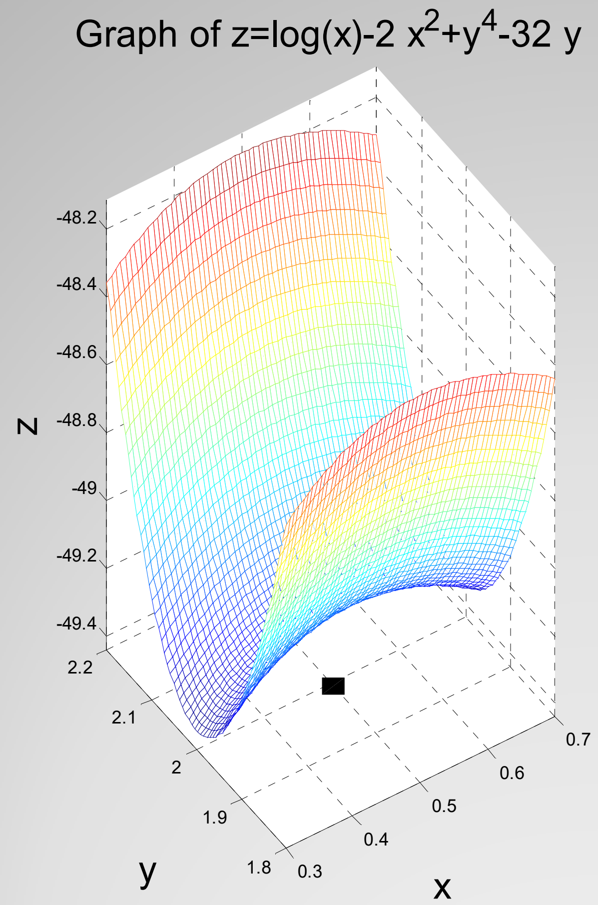
$$Hf(x,y)=\begin{pmatrix} -\frac{1}{x^2}-4 & 0 \\ 0 & 12y^2 \end{pmatrix} \Rightarrow Hf(1/2,2)=\begin{pmatrix} -8 & 0 \\ 0 & 48 \end{pmatrix}$$

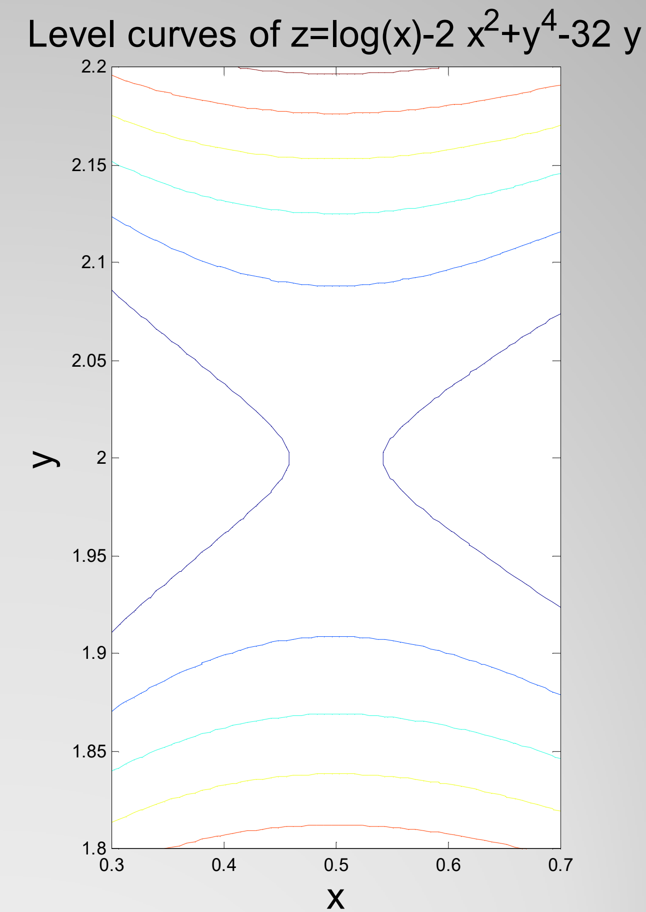
The diagonal matrics has eigenvalues equal to -8 and 48

Hence the hessian matrix in the critical point is indefinite then:

(1/2,2) is a SADDLE POINT of f

From the graph and level curves of f in an r-Ball about (1/2,2) the saddle point can be observed!





Ex10: Determine the local max and min points of the following function: $z = x_1^{\ 3} + x_2^{\ 2} + 2x_1x_2 - (x_3 - 1)^2$

The **critical points** are given by the solutions of the following system:

$$\begin{cases} z_{x_1} = 3x_1^{\ 2} + 2x_2 = 0 \\ z_{x_2} = 2x_2 + 2x_1 = 0 \\ z_{x_3} = -2(x_3 - 1) = 0 \end{cases} \Rightarrow \begin{cases} 3x_1^{\ 2} + 2x_2 = 0 \\ x_2 = -x_1 \\ x_3 = 1 \end{cases} \Rightarrow \begin{cases} 3x_1^{\ 2} - 2x_1 = 0 \\ x_2 = -x_1 \\ x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1(3x_1 - 2) = 0 \\ x_2 = -x_1 \\ x_3 = 1 \end{cases} \Rightarrow$$

$$I \begin{cases} x_1 = 0 \\ x_2 = 0 \Rightarrow P = (0,0,1) \\ x_3 = 1 \end{cases}$$

$$II \begin{cases} (3x_1 - 2) = 0 \\ x_2 = -x_1 \\ x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 2/3 \\ x_2 = -2/3 \Rightarrow Q = (2/3, -2/3, 1) \\ x_3 = 1 \end{cases}$$

The Hessian matrix is given by:

$$Hf(\underline{x}) = \begin{pmatrix} 6x_1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \text{ so that } Hf(P) = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \text{ while } Hf(Q) = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

We use MatLab to find the eigenvalues of the two matrices:

```
>> eig([0 2 0;2 2 0;0 0 -2])

ans =

  -2.0000

  -1.2361

   3.2361

>> eig([4 2 0;2 2 0;0 0 -2])

ans =

  -2.0000

   0.7639

   5.2361
```

P and Q are SADDLE POINTS of *f*

Homeworks

EX 1.6

Determine the local max and min points of the following functions (you can use MatLab to calculate the eigenvalues).

$$(1)\; y = -2x_1^2 - 4x_2^2 - x_3^2 + 4x_1 + x_3 - 6$$

$$(2)\; z = \frac{1}{2}x^2 - \ln x + y^2 - 2y$$

$$(3)\; y = (x_1 - 2)^3 + x_2{}^3 + 2x_3{}^2 - 2x_3x_2$$

$$(4)\; z = \frac{x^3}{3} + x^2 - 3x + y^2$$

$$(5)\; y = 2x_1{}^2 - 2x_1x_2 + x_2{}^2 - 6x_2 + 1 + x_3^3 - 3x_3$$

NOTICE THAT:

in general an unconstrained optimization problem can be difficult to be solved.

The complexity of the problem-solution depends on several factors such as:

- THE NUMBER OF VARIABLES,
- the ANALYTICAL FORM of the GIVEN FUNCTION,
- the impossibility to conclude when the HESSIAN IS SEMI-DEFINITE etc.

Such cases will be attached by using MatLab!