International Finance and Economics
Dept. of Economics and Law

Mathematical methods for economics and finance

Prof. Elisabetta Michetti

PART 5 - Theory

THE CONSTRAINED PROBLEMS

Constrained optimization problems arise quite naturally in economics and finance. Some examples are the following:

- Max utility given the available income
- Min portofolio risk given an expected return
- Max profits given the technology and the production costs

Such kinds of problems can be formalized in the following way.

Let $f : A \subseteq R^n \to R$ and $g_i, h_j : \mathrm{B} \subseteq R^n \to R \quad (i = 1, \ldots, m;\ j = 1, \ldots, k)$.

we want to find local max or min pts of $f(\underline{x})$ under the constraints

$$g_1(\underline{x}) = 0, \ldots, g_m(\underline{x}) = 0$$

$$h_1(\underline{x}) \geq 0, h_2(\underline{x}) \geq 0, \ldots, h_k(\underline{x}) \geq 0$$

THE PROTOTYPE PROBLEM (*):

Maximize (or minimize) $f(x_1,\ldots,x_n)$

under the following constraints:

$$\begin{cases} g_1(\underline{x})=0,\ldots,g_m(\underline{x})=0 \\ h_1(\underline{x})\geq 0, h_2(\underline{x})\geq 0,\ldots,h_k(\underline{x})\geq 0 \end{cases}$$

f is the **objective function**,

g_i (i=1,2,…m) are the **equality constraints**

h_j (j=1,2,…,k) are the **inequality constraints;** in applications most common inequality constraints are the **non-negativity constraints**:

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

EX1 • Utility maximization problem

Consider 3 commodities (1,2,3) whose prices are respectively 2, 5, 4 euro. It is also given a linear utility function $U=x_1+2x_2+0.5x_3$. The available income is 80 euro.

The consumer wants to **maximize its utility by taking into account the budget constraint** (the available income) and the fact that the quantities of each good cannot be negative.

FORMALIZATION:
3 variables,
4 inequality constraints

Max $x_1 + 2x_2 + 0.5x_3$

under the following constraints:

$$\begin{cases} 2x_1 + 5x_2 + 4x_3 \le 80 \\ x_1 \ge 0 \\ x_2 \ge 0 \\ x_3 \ge 0 \end{cases}$$

EX2

• Profit maximization problem

Consider a firm in a competitive market producing the output y by using 4 inputs and let the Cobb-Douglas production function be given by: $y=10x_1^{0.2}x_2^{0.3}x_3^{0.1}x_4^{0.9}$

Let p=5 be the unit price of good y. The cost of each input is given respectively by 2,3,2,4 euro. The profits are then given by $\pi=50x_1^{0.2}x_2^{0.3}x_3^{0.1}x_4^{0.9}-2x_1-3x_2-2x_3-4x_4$. **The firm wants to maximize the profits by paying for the inputs at most a cost equal to 400.**

FORMALIZATION:

4 variables,

5 inequality constraints

Max $50x_1^{0.2}x_2^{0.3}x_3^{0.1}x_4^{0.9}-2x_1-3x_2-2x_3-4x_4$

under the following constraints:

$$\begin{cases} 2x_1+3x_2+2x_3+4x_4\le 400 \\ x_1\ge 0 \\ x_2\ge 0 \\ x_3\ge 0 \\ x_4\ge 0 \end{cases}$$

EX3 • Portfolio risk minimization problem

Consider 3 assets (1,2,3) in portfolio and let x_i be fraction of asset i in the portfolio. The portfolio risk depends on the fractions of each asset in the portfolio as follows:

$R_p=0.2x_1^2-0.1x_1x_2+0.5x_2^2+0.1x_3^2$. The expected return of each asset is respectively 0.05, 0.1, 0.03. The investor wants to determine how much to invest in each asset in order to **minimize risk while assuring an expected portfolio return equal to 0.07**.

FORMALIZATION:

3 variables,

3 inequality constraints

2 equality constraints

$$\text{Min } 0.2x_1^2 - 0.1x_1x_2 + 0.5x_2^2 + 0.1x_3^2$$

under the following constraints:

$$\begin{cases} 0.05x_1 + 0.1x_2 + 0.03x_3 = 0.07 \\ x_1 + x_2 + x_3 = 1 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{cases}$$

Def: absolute (or global) CONSTRAINED maximum pt and absolute (or global) CONSTRAINED minimum pt

Consider the prototype problem (*) and let

$$E = \{\underline{x} \in A : g_i(\underline{x}) = 0 \text{ and } h_j(\underline{x}) \geq 0, i = 1,\ldots,m, j = 1,\ldots,k \}$$

$\underline{x}^*$ is an ABSOLUTE CONSTRAINED MAX point if

$$f(\underline{x}^*) \geq f(\underline{x}) \qquad \forall \underline{x} \in E$$

$\underline{x}^*$ is an ABSOLUTE CONSTRAINED MIN point if

$$f(\underline{x}^*) \leq f(\underline{x}) \qquad \forall \underline{x} \in E$$

Notice: If a point is an absolute CONSTRAINED max then there are no points in the domain of the constraints at which f takes a larger value

Def: relative (or local) CONSTRAINED maximum pt and relative (or local) CONSTRAINED minimum pt

Consider the prototype problem (*) and let

$$E = \{\underline{x} \in A : g_i(\underline{x}) = 0 \text{ and } h_j(\underline{x}) \geq 0, i = 1, \ldots, m, j = 1, \ldots, k \}$$

$\underline{x}^*$ is a RELATIVE CONSTRAINED MAX point if

$$\exists\, B(\underline{x}^*, r) : \; f(\underline{x}^*) \geq f(\underline{x}) \quad \forall \underline{x} \in B(\underline{x}^*, r) \cap E$$

$\underline{x}^*$ is an RELATIVE CONSTRAINED MIN point if

$$\exists\, B(\underline{x}^*, r) : \; f(\underline{x}^*) \leq f(\underline{x}) \quad \forall \underline{x} \in B(\underline{x}^*, r) \cap E$$

Notice: If a point is a CONSTRAINED local max then there are no nearby points verifying the constraints at which f takes a larger value

The **main goal** of this section it to give an answer to the following **problem**.

Let y=f(x) be a function of several variables, we want to determine its local maximum and local minimum points under the constraints

$$g_1(\underline{x}) = 0, \ldots, g_m(\underline{x}) = 0$$

AND/OR

$$h_1(\underline{x}) \geq 0, h_2(\underline{x}) \geq 0, \ldots, h_k(\underline{x}) \geq 0$$

CASE 1: Equality constraints

Consider $f : A \subseteq R^n \to R$ and $g_i : B \subseteq R^n \to R \;\; (i = 1, \ldots, m) \quad m < n$

determine **max** and **min** pts of f

restricted to the set $E = \{\underline{x} \in A : g_1(\underline{x}) = 0, \ldots, g_m(\underline{x}) = 0\}$

The set $\boldsymbol{E}$ with points $\underline{\boldsymbol{x}} \in A \subseteq \boldsymbol{R}^n$

satisfying the $\boldsymbol{m}$ equations

$$\begin{cases} g_1(x_1, \ldots, x_n) = 0 \\ g_2(x_1, \ldots, x_n) = 0 \\ \vdots \\ g_m(x_1, \ldots, x_n) = 0 \end{cases}$$

is the **FEASIBLE SET**

Def: Jacobian matrix of $\underline{g}$

The Jacobian matrix associated to the equality constraints is given by the matrix J (mxn) whose row vectors are the m gradients of the functions defining the equality constraints:

$$J\underline{g}(\underline{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\underline{x}) & \cdots & \frac{\partial g_1}{\partial x_n}(\underline{x}) \\ \vdots & \vdots & \vdots \\ \frac{\partial g_m}{\partial x_1}(\underline{x}) & \cdots & \frac{\partial g_m}{\partial x_n}(\underline{x}) \end{pmatrix}$$

Def: Regular point

A point $\underline{x}^* \in A \subseteq R^n$ s.t. $g_i(\underline{x}^*) = 0 \;\; (i = 1, \ldots, m)$

Is called REGULAR POINT if:

The rank of the Jacobian matrix $J\underline{g}(\underline{x}^*)$ is equal to m

We recall that the **rank of a matrix J (mxn) with m<n is equal to m** if there exists an (mxm) matrix from J (obtained with the common elements of m-rows and m-columns) having the determinant different from zero.

EX4

Consider the following problem:

max/min $y = f(x_1, x_2, x_3)$ such that $\begin{cases} x_1^2 + x_2^2 = 1 \\ x_1 + x_3 = 1 \end{cases}$.

The constraints can be rewritten as $\begin{cases} x_1^2 + x_2^2 - 1 = 0 \\ x_1 + x_3 - 1 = 0 \end{cases}$

and the Jacobian matrix is given by: $\mathrm{J}\underline{\mathrm{g}}(x_1, x_2, x_3) = \begin{pmatrix} 2x_1 & 2x_2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

The rank of J is less then 2 iff $x_1 = x_2 = 0$.

Since any point with $x_1 = x_2 = 0$ is not feasible (as it violates the first constraint), then all feasible points are regular points!

HOMEWORKS

EX 1.1

Determine the Jacobian matrix associated to the following problems and estabilish if there are feasible points that are not regular points.

$$1)\ \max/\min y = f(x_1, x_2, x_3, x_4) \text{ such that } \begin{cases} x_1^{\,3} + x_2^{\,3} = x_4 \\ x_1 + 2x_2 - x_3 + 3x_4 = 5 \end{cases}.$$

$$2)\ \max/\min y = f(x_1, x_2, x_3) \text{ such that } \begin{cases} x_3 = 0 \\ -x_1 - 2x_2 + x_3 = 5. \\ 2x_1^{\,2} + x_2 - 1 = 0 \end{cases}$$

First order conditions: THEOREM

Let $f, g_1, \ldots, g_m$ be C^1 functions

and $\underline{x}^* \in E = \{\underline{x} \in A \subseteq R^n : g_i(\underline{x}) = 0, i = 1, \ldots, m\}$ a regular point.

If $\underline{x}^*$ is a constrained local max or min pt of f on E then

there exist m real numbers $\lambda_1^*, \ldots, \lambda_m^*$ (Lagrange multipliers)

such that $(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$ is a critical point

of the following LAGRANGIAN FUNCTION

$$L(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \ldots - \lambda_m g_m(\underline{x})$$

According to the previous Theorem the optimum constrained points must be searched within the critical points of the Lagrangian function.

Anyway it is very important to observe that **the previous condition is only necessary**: a constrained max or min point is a critical point of function L but **a critical point of L may be a constrained max pt, a constrained min pt, or neither.**

Notice also that in order to find the critical points of a problem with n variables and m equality constraints a system with n+m equations must be solved!

EX5 From EX 4 consider the following problem:

$$\max/\min\ y = x_1 + x_2 + x_3 \text{ such that } \begin{cases} x_1^{\ 2} + x_2^{\ 2} = 1 \\ x_1 + x_3 = 1 \end{cases}.$$

1) First of all we define the Lagrangian function:

$$L = x_1 + x_2 + x_3 - \lambda_1 (x_1^{\ 2} + x_2^{\ 2} - 1) - \lambda_2 (x_1 + x_3 - 1).$$

2) Secondly we determine the 5 partial derivatives, w.r.t. $x_1, x_2, x_3, \lambda_1, \lambda_2$ and we cosntruct the system by imposing that all these partial derivatives are equal to zero.

$$\begin{cases} L_{x_1} = 1 - 2\lambda_1 x_1 - \lambda_2 = 0 \\ L_{x_2} = 1 - 2\lambda_1 x_2 = 0 \\ L_{x_3} = 1 - \lambda_2 = 0 \\ L_{\lambda_1} = -x_1^{\ 2} - x_2^{\ 2} + 1 = 0 \\ L_{\lambda_2} = -x_1 - x_3 + 1 = 0 \end{cases}$$

...EX5

3) Solve the system (if possible!)

$$\begin{cases} 1-2\lambda_1 x_1-\lambda_2=0 \\ 1-2\lambda_1 x_2=0 \\ \lambda_2=1 \\ -x_1^2-x_2^2+1=0 \\ -x_1-x_3+1=0 \end{cases} \Rightarrow \begin{cases} 2\lambda_1 x_1=0 \Rightarrow \lambda_1=0, x_1=0 \\ 1-2\lambda_1 x_2=0 \\ \lambda_2=1 \\ -x_1^2-x_2^2+1=0 \\ -x_1-x_3+1=0 \end{cases} \Rightarrow I\begin{cases} \lambda_1=0 \\ 1=0 \\ \lambda_2=1 \\ -x_1^2-x_2^2+1=0 \\ -x_1-x_3+1=0 \end{cases} \Rightarrow \text{NO SOLUTION!}$$

$$\Rightarrow II\begin{cases} x_1=0 \\ 1-2\lambda_1 x_2=0 \\ \lambda_2=1 \\ x_2=\pm 1 \\ x_3=1 \end{cases} \Rightarrow II.1\begin{cases} x_1=0 \\ \lambda_1=1/2 \\ \lambda_2=1 \\ x_2=+1 \\ x_3=1 \end{cases} \cup II.2\begin{cases} x_1=0 \\ \lambda_1=-1/2 \\ \lambda_2=1 \\ x_2=-1 \\ x_3=1 \end{cases}$$

4) The Lagrangian has 2 critical points: P=(0,1,1,1/2,1) and Q=(0,-1,1,-1/2,1);

such points are regular points;

hence the constrained optimization problem can have up to two optimal solutions.

Each point (0,1,1) and (0,-1,1) can be a (loc) constrained max, a (loc) constrained min or noone of them!

EX6

Let x, y and z be quantities consumed of goods X, Y and Z.

The utility of the consumer is described by the following function:

$$U=zy+y^2+2x.$$

Assume that the prices of the three goods are respectively 3,4,1 and that the available income is equal to 50. The consumer wants to maximize its utility by taking into account the budget constraint and using all the available income.

We formalize the problem and determine the points that are candidates to be a solution to the problem.

...EX6 (for the moment we do not consider the non-negativity constraints) $\max zy + y^2 + 2x : \{3x + 4y + z = 50$

$$\Rightarrow L = zy + y^2 + 2x - \lambda(3x + 4y + z - 50)$$

$$\Rightarrow \begin{cases} L_x = 2 - 3\lambda = 0 \\ L_y = z + 2y - 4\lambda = 0 \\ L_z = y - \lambda = 0 \\ L_\lambda = 3x + 4y + z - 50 = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 2/3 \\ z = -2y + 8/3 \\ y = 2/3 \\ 3x + 4y + z - 50 = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda = 2/3 \\ z = -2y + 8/3 \\ y = 2/3 \\ x = \dfrac{-4y - z + 50}{3} \end{cases} \Rightarrow \begin{cases} \lambda = 2/3 \\ z = 4/3 \\ y = 2/3 \\ x = \dfrac{138}{9} \end{cases}$$

The point P=(138/9,2/3,4/3,2/3) is a critical point of L and the quantities x=138/9,y=2/3,z=4/3 (that are all positive) can be the optimal solution of the constrained utility maximization problem!

HOMEWORKS

EX 1.2

Determine the points that are candidates to be local constrained max or min of the following problems. Verify also if the obtained points are regular points.

$$1)\ y = 2x_1 - x_2 + 3x_3 \text{ such that } \begin{cases} 2{x_3}^2 + {x_1}^2 = 4 \\ x_1 - x_3 = 1 \end{cases}.$$

$$2)\ z = x^2 + 2y^2 \text{ such that } 2x + 6y = 30.$$

HOMEWORKS

EX 1.3

Let x, y and z be the fraction invested in the risky assets X, Y and Z in a portfolio.

Let the portofolio risk be described by the following law: $R=x^2-xy+y^2+2z^2$.

Assume that in the market it is possible to invest or to borrow (i.e. the three variable can be both positive or negative). We want to determine a possible combination of fraction x,y,z invetsed in each risky asset to minimize the portfolio risk. Formalize the problem and determine the points that are candidates to be a solution to the problem. Verify also if the obtained points are regular points.

EX 1.4

Let x and y be the quantities of labour and capital used in production. Assume a Cobb-Douglas production function given by $Q=10x^{0.7}y^{0.3}$.

The unitary cost of labour is 2 and the unitary cost of capital is 8. The total cost that the firm wants to support is 200.

The firm wants to maximize the production under the total cost constraint. Formalize the problem and determine the points that are candidates to be a solution to the problem. Are the obtained points regular points?

Def: Hessian bordered matrix

Let $f, g_1, \ldots, g_m$ be C^2 functions, and let $(\underline{x}^*, \underline{\lambda}^*)$ be a critical point of the Lagrangian function.

The BORDERED HESSIAN MATRIX asscoiated to point $(\underline{x}^*, \underline{\lambda}^*)$ is:

$$HL(\underline{x}^*, \underline{\lambda}^*) = \begin{pmatrix} \underline{0} & J\underline{g}(\underline{x}^*) \\ J\underline{g}(\underline{x}^*)^T & H_{\underline{x}}L(\underline{x}^*, \underline{\lambda}^*) \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} & \frac{\partial^2 L}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_n \partial x_n} \end{pmatrix}$$

Notice that HL is a symmetric square matrix (m+n x m+n).

Second order conditions: THEOREM

Let $f, g_1, \ldots, g_m$ be C^2 functions

and let $(\underline{x}^*, \underline{\lambda}^*)$ be a critical point of the associated Lagrangian function

(1) If the bordered Hessian of L at $(\underline{x}^*, \underline{\lambda}^*)$ is a negative definite matrix

on the set $\left\{\underline{v} \neq \underline{0} : J\underline{g}(\underline{x}^*)\underline{v} = \underline{0}\right\}$

then $\underline{x}^*$ is a relative constrained MAX pt of f on E

(2) If the bordered Hessian of L at $(\underline{x}^*, \underline{\lambda}^*)$ is a positive definite matrix

on the set $\left\{\underline{v} \neq \underline{0} : J\underline{g}(\underline{x}^*)\underline{v} = \underline{0}\right\}$

then $\underline{x}^*$ is a relative constrained MIN pt of f on E

PRACTICAL RULE: with constrained bordered Hessian the rule of eigenvalues cannot be used to conclude about the definition.

To verify the sufficient conditions the following procedure holds for the bordered Hessian matrix using the determinants

DEF : the leading principal minors of order k of matrix A is given by the DETERMINANT of the submatrix of A composed by the FIRST k-ROWS AND k-COLOMNS

1) If the sign of |HL| is $(-1)^n$ and the last n-m leading principal minors alternate in sign then the candidate point is a constrained max point

2)If the last n-m leading principal minors have all the sign of $(-1)^m$ then the candidate point is a constrained minimum point

EX7

Maximize the following function

$$f(x, y, z) = x^2 y^2 z^2$$

with the constraint $x^2 + y^2 + z^2 = 3$

and $x, y, z \geq 0$

For the moment we do not consider the inequality constraints but we will check if the solution verifies such a requirement.

Lagrangian function

$$L(x, y, z, \lambda) = x^2 y^2 z^2 - \lambda(x^2 + y^2 + z^2 - 3)$$

…EX7

$$\begin{cases} \dfrac{\partial L}{\partial x} = 2xy^2z^2 - 2\lambda x = 0 \\ \dfrac{\partial L}{\partial y} = 2x^2yz^2 - 2\lambda y = 0 \\ \dfrac{\partial L}{\partial z} = 2x^2y^2z - 2\lambda z = 0 \\ \dfrac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 3 = 0 \end{cases}$$

The solution is:

$$x^2 = y^2 = z^2 = \lambda = 1$$

And considering the non negativity constraints:

$$x^* = y^* = z^* = \lambda^* = 1$$

$$HL(x,y,z,\lambda) = \begin{pmatrix} 0 & 2x & 2y & 2z \\ 2x & 2y^2z^2 - 2\lambda & 4xyz^2 & 4xy^2z \\ 2y & 4xyz^2 & 2x^2z^2 - 2\lambda & 4x^2yz \\ 2z & 4xy^2z & 4x^2yz & 2x^2y^2 - 2\lambda \end{pmatrix}$$

…EX7

in point $x^* = y^* = z^* = \lambda^* = 1$ we get

$$\mathrm{HL}(x^*, y^*, z^*, \lambda^*) = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{pmatrix}$$

$$|D_3| = \begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 0 \end{vmatrix} = 32 > 0 \qquad |D_4| = |HL| = -192 < 0$$

and sign of $|HL|$ is $(-1)^3 = -1 < 0$

$$\Downarrow$$

x=y=z=1 is a constrained max point

EX8

Consider the problem:

$\max 5x_1 + 2x_2 - x_3$ with constraints $x_1x_2 = 3; \quad x_1x_3 = 1$

We have n=3 and m=2.

The Lagrangian function is:

$$L(\underline{x}, \underline{\lambda}) = 5x_1 + 2x_2 - x_3 + \lambda_1(3 - x_1x_2) + \lambda_2(1 - x_1x_3)$$

The first order condition requires

$$5 - \lambda_1x_2 - \lambda_2x_3 = 0$$
$$2 - \lambda_1x_1 = 0$$
$$-1 - \lambda_2x_1 = 0$$
$$3 - x_1x_2 = 0$$
$$1 - x_1x_3 = 0$$

So that two solutions are obtained:

P=(1,3,1,2,-1) and Q=(-1,-3,-1,2,1).

EX8

The border Hessian is given by :

$$H=\begin{pmatrix} 0 & 0 & x_2 & x_1 & 0 \\ 0 & 0 & x_3 & 0 & x_1 \\ x_2 & x_3 & 0 & -\lambda_1 & -\lambda_2 \\ x_1 & 0 & -\lambda_1 & 0 & 0 \\ 0 & x_1 & -\lambda_2 & 0 & 0 \end{pmatrix}$$

And in order to have a constrained maximum it is necessary that the sign of the determinant of H is equal to -1. This condition holds iff the point P is considered. Furthermore, since n-m=3-2=1 then it is the only determinant that we have to consider.

Then we can conclude that P is the local constrained maximum point!

Notice: you can calculate the determinants with **MatLab**! 1) define the matrix, ex H, 2) calculate the determinant of H using the command **det(H)**.

HOMEWORKS

EX 1.5

Determine the constrained local max and min of the following problems:

$$1)\, z = -xy : x^2 + y^2 - 4 = 0$$

$$2)\, z = x^2 + 2y^2 : x^2 + y^2 - 1 = 0$$

$$3)\, z = x^3 + y^3 - 3xy : y = x$$

$$4)\, y = 2x_1 + 3x_2 + x_3 : x_1^2 + 2x_2^{\,2} = 4$$

EX 1.6

Let x, y and z be the quantities of labour, capital and materials used in production. Assume a production function given by Q=xyz. The unitary cost of each input is respectively 2,3,1.

The firm wants to produce 200 units of goods by paying the minimum total cost. Formalize the problem and solve it. (Notice that only not negative values must be considered)

CASE 2: Inequality constraints

Consider $f : A \subseteq R^n \to R$ and $h_j : B \subseteq R^n \to R \ \ (j = 1, \dots, k)$

determine **max** and **min** of f

restricted to the set $E = \left\{ \underline{x} \in A : h_j(\underline{x}) \geq 0, j = 1, \dots, k \right\}$

The set $\boldsymbol{E}$ with points $\underline{\boldsymbol{x}} \in A \subseteq \boldsymbol{R}^n$

satisfying the $\boldsymbol{k}$ inequalities

$$\begin{cases} h_1(x_1, \dots, x_n) \geq 0 \\ h_2(x_1, \dots, x_n) \geq 0 \\ \vdots \\ h_k(x_1, \dots, x_n) \geq 0 \end{cases}$$

is the **FEASIBLE SET**

Notice that:

A constraint $h_j \leq 0$ can always be written in the form $-h_j \geq 0$

A maximization problem can always be changed in a minimization problem by considering that:

$$\max_E(f) = -\min_E(-f)$$

Simple case:

2 variables, one constraint

$$\min_{x,y} f(x,y)$$

such that $h(x,y) \geq 0$

First order condition – THEOREM

Let $f, h \in C^1$ and (x^*, y^*) be a local constrained min of f on E

Furthermore if $h(x^*, y^*) = 0$, then $\nabla h(x^*, y^*) \neq \underline{0}$

Define the Lagrangian function as

$$L(x, y, \mu) = f(x, y) - \mu h(x, y)$$

Then $\exists$ **a multiplier** μ^* **such that**

$$(*)\quad \begin{cases} \dfrac{\partial L}{\partial x}(x^*, y^*, \mu^*) = 0 \\ \dfrac{\partial L}{\partial y}(x^*, y^*, \mu^*) = 0 \\ \mu^* \cdot h(x^*, y^*) = 0 \\ \mu^* \geq 0 \\ h(x^*, y^*) \geq 0 \end{cases}$$

$(*)$ are the **Kuhn - Tucker conditions**

EX9

$$\min_{(x,y)} 2y - x^2 \text{ s.t. } x^2 + y^2 \le 1$$

The constraint can be rewritten as: $-x^2 - y^2 + 1 \ge 0$

$$L(x, y, \mu) = 2y - x^2 - \mu(-x^2 - y^2 + 1)$$

The **Kuhn-Tucker** first order conditions are given by:

$$\frac{\partial L}{\partial x} = -2x + 2\mu x = 0$$

$$\frac{\partial L}{\partial y} = 2 + 2\mu y = 0$$

$$\mu(-x^2 - y^2 + 1) = 0$$

$$\mu \ge 0$$

$$-x^2 - y^2 + 1 \ge 0$$

and consequently:

$\mu^* = 1, x^* = 0, y^* = -1$

(notice that the gradient of h at this point is not zero)

$\Rightarrow$ It can be the a constrained min point!

EX10

We consider the consumer constrained maximization problem with two goods and a utility function U=2xy. Let the price of good x be equal to 2 and the price of good y be equal to 1. The available income is equal to 20.

The initial problem is:

$$\max_{x,y} 2xy$$

$$: 2x + y \leq 20, x \geq 0, y \geq 0$$

The equivalent problem is:

$$\min_{x,y} -2xy$$

$$: -2x - y + 20 \geq 0, x \geq 0, y \geq 0$$

$$\Rightarrow L = -2xy - \mu(-2x - y + 20)$$

...EX10

The KT conditions are

(without considering non negativity constraints):

$$\begin{cases} \dfrac{\partial L}{\partial x} = -2y + 2\mu = 0 \\ \dfrac{\partial L}{\partial y} = -2x + \mu = 0 \\ \mu(-2x - y + 20) = 0 \\ \mu \geq 0 \\ -2x - y + 20 \geq 0 \end{cases} \Rightarrow \begin{cases} y = \mu \\ 2x = \mu \\ \mu(-2\mu + 20) = 0 \\ \mu \geq 0 \\ -2x - y + 20 \geq 0 \end{cases} \Rightarrow \begin{cases} y = 10 \\ x = 5 \\ \mu = 10 \\ \mu \geq 0 \\ -2x - y + 20 \geq 0 \end{cases}$$

The point **P=(5,10,10)** is candidate to be a solution of the constrained maximization problem (notice that in such a point the non negativity constraints hold). Since μ≠0 the constraint holds with the equality in point P.

HOMEWORKS

EX 1.7

Determine the set of points that are candidates to be a solution of the following problems:

$$1)\ \min xy : x + 2y \geq 4$$

$$2)\ \max 2x + 3y - 1 : y - x^2 + 1 \geq 0$$

$$3)\ \min 2x + 4y : xy = 10$$

EX 1.8

Consider the following production function depending on two inputs: Q=10xy. The unitary cost of each inputs are respectively 1 and 2. The firm wants to determine the quantities of the inputs in order to maximize the production by having a cost at most equal to 50. (Notice that only not negative values must be considered)

In order to determine if the candidate point is the minimum constrained point observe that two cases may occur:

(a) If at point (x*,y*) the constraint holds with equality, that is h(x*,y*)=0 (that is μ*>0) then the problem can be considered as a problem with equality constraint and the following condition holds.

If the following matrix

$$HL(x^*, y^*, \mu^*) = \begin{pmatrix} 0 & \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} \\ \frac{\partial h}{\partial y} & \frac{\partial^2 L}{\partial y \partial x} & \frac{\partial^2 L}{\partial y^2} \end{pmatrix}$$

is such that $|D(x^*, y^*, \mu^*)| < 0$,

then (x*,y*) is a constrained minimum point!

(b) **If at point (x*,y*) the constraint holds with strict inequality**, that is h(x*,y*)>0 (that is μ*=0) then the problem can be considered as an unconstrained problem where (x*,y*) is an interior point and the following condition holds.

If the following matrix

$$H(x^*, y^*) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

is positive definite

then (x*,y*) is a constrained minimum point!

EX11

A firm can produce each week the following quantity Q of paper depending on the number of hours worked (N) and on the invested capital (K): $Q = 30N^2K$.

The firm wants to produce al least 3000 kg of paper each week.

Determine the values of N and K such that the cost for the inputs given by $C = 15N + 720K$ are minimized.

THE PROBLEM: $\min 15N + 720K$

such that $30N^2K \geq 3000$

that is $N^2K - 100 \geq 0$

$(N \geq 0, K \geq 0)$

...EX11

1. Write the Lagrangian

$$L = 15N + 720K - \mu(N^2K - 100)$$

2. Solve the KT conditions

$$\begin{cases} \dfrac{\partial L}{\partial N} = 15 - 2\mu NK = 0 \\ \dfrac{\partial L}{\partial K} = 720 - \mu N^2 = 0 \\ \mu(N^2K - 100) = 0 \\ \mu \geq 0 \\ N^2K - 100 \geq 0 \end{cases} \Rightarrow \begin{cases} 15 - 2\mu NK = 0 \\ \mu = 720 / N^2 \\ K = 100 / N^2 \\ \mu > 0 \\ N^2K - 100 \geq 0 \end{cases} \Rightarrow \begin{cases} 15 - 144000 / N^3 = 0 \\ \mu = 720 / N^2 \\ K = 100 / N^2 \\ \mu > 0 \\ N^2K - 100 \geq 0 \end{cases} \Rightarrow \begin{cases} N = 21.253... > 0 \\ \mu = 1.594... > 0 \\ K = 0.2214... > 0 \\ \mu > 0 \\ N^2K - 100 = 0 \end{cases}$$

$$\Rightarrow P = (21.253..., 0.2214..., 1.594...)$$

...EX11

3. Since in such a point the constraint is verified with the equality, we consider the bordered Hessian matrix

$$\mathrm{HL}(N, K, \mu) = \begin{pmatrix} 0 & 2NK & N^2 \\ 2NK & -2\mu K & -2\mu N \\ N^2 & -2\mu N & 0 \end{pmatrix}$$

$$|\,\mathrm{HL}(21.253..., 0.2214..., 1.594...)\,| = \left| \begin{pmatrix} 0 & 9.41 & 451.69 \\ 9.41 & -0.7 & -67.75 \\ 451.69 & -67.75 & 0 \end{pmatrix} \right| < 0$$

$$\Rightarrow (N^*, \mathrm{K}^*) \text{ is a constrained minimum point!}$$

Hnce the firm must buy 21.253 and 0.2214 quantities of the two inputs to produce 3000 Kg of paper each week at the minimum cost.

HOMEWORKS

EX 1.8

Determine the nature of the candidate points in EX 1.8

EX 1.9

Consider a consumer who wants to maximize the utility function $U=xy^3$. The price of the two goods are 2 and 3 respectively and the maximum amount of income is 40. Solve the constrained problem.

EX 1.10

An agent wants to buy red and yellow flowers by paying at most 100 euro. Each yellow flower costs 10 euro while each red flower costs 5 euro. Let x (y) be the number of yellow (red) flower and consider the following preference function over the two variables $P=x^2+y$. Solve the problem of the opitmum choice for the agent (that is the maximum satisfaction under the budget constraint).

The more general case can consider

- several inequality constraints
- several equality constraints

The necessary first order and the sufficient second order conditions can be generalized, anyway the analytical procedure to solve the problem can be complex.

NOTICE THAT:

in general constrained optimization problem can be difficult to be solved.

The complexity of the problem-solution depends on several factors such as:

- THE NUMBER OF VARIABLES,
- THE NUMBER OF CONSTRAINTS
- the ANALYTICAL FORM of the GIVEN FUNCTION,
- the impossibility to conclude when the SUFFICIENT CONDITIONS does not hold..

Such cases will be attached by using MatLab!