International Finance and Economics
Dept. of Economics and Law

Mathematical methods for economics and finance

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PART 6 - MatLab

Find minimum pts of constrained functions

In MatLab it is possible to use the **optimtool** to solve a minimization problem of an objective function with several constraints.

The function to be used is **fmincon** AND THE STANDARD MATLAB FORMALIZATION MUST BE CONSIDERED!

FORMALIZATION OF THE PROBLEM (standard MatLab form)

$$\min_{\underline{x}} f(\underline{x}) \text{ such that}$$

$$\begin{cases} A\underline{x} \leq \underline{b} \\ Aeq\,\underline{x} = \underline{beq} \\ \underline{lb} \leq \underline{x} \leq \underline{ub} \\ \underline{c}(\underline{x}) \leq \underline{0} \\ \underline{ceq}(\underline{x}) = \underline{0} \end{cases}$$

Arguments specifications

$$\min_{\underline{x}} f(\underline{x})$$

Function f is the objective function and MatLab can find the minimum (constrained) points. Hence if we have to find the **maximum (constrained) point of a function f** the problem must be rewritted as a **minimization (constrained) problem of function –f!**

EX1 max -3x+yz can be written as min 3x-yz!

Following the usual instruction they can be saved in MatLab!

RECALL:

-use the function handle @(x) where x is a vector;

-define the function by using the variables x(1), x(2),…, x(n);

-don't use spaces and don't use the punctual operators.

Arguments specifications

The constraints can be linear or non linear!

LINEAR CONSTRAINTS (inequality):

$A\underline{x} \le \underline{b}$ Is a **linear inequality** constraint where **A is the matrix of the coefficients** while **b is the vector of the constants**.

EX2 the following linear inequality constraints

$$\begin{cases} 3x+2y \le 2z+2 \\ 4x-6y \ge -10 \\ x-y-2z \le 8 \\ 7 \ge 3y-z \end{cases}$$

Must be written in the standard form

$$\Rightarrow \begin{cases} 3x+2y-2z \le 2 \\ -4x+6y \le 10 \\ x-y-2z \le 8 \\ 3y-z \le 7 \end{cases}$$

Hence A and b are given by

$$A = \begin{pmatrix} 3 & 2 & -2 \\ -4 & 6 & 0 \\ 1 & -1 & -2 \\ 0 & 3 & -1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 2 \\ 10 \\ 8 \\ 7 \end{pmatrix}$$

Following the usual instruction they can be saved in MatLab!

Arguments specifications

LINEAR CONSTRAINTS (equality):

$$Aeq\,\underline{x} = \underline{beq}$$

Is a **linear equality** constraint where **Aeq is the matrix of the coefficients** while **beq is the vector of the constants**.

EX3 the following linear equality constraints

$$\begin{cases} 2x_1 + 3x_2 = 4 - x_4 \\ -x_2 + x_1 + 2x_3 + 2 = 0 \end{cases}$$

Must be written in the standard form

$$\Rightarrow \begin{cases} 2x_1 + 3x_2 + x_4 = 4 \\ x_1 - x_2 + 2x_3 = -2 \end{cases}$$

Hence Aeq and beq are given by

$$Aeq = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}, \underline{beq} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

Following the usual instruction they can be saved in MatLab!

Arguments specifications

LINEAR CONSTRAINTS (lower and upper bounds):

$$\underline{lb} \le \underline{x} \le \underline{ub}$$

Are the bounds (lower and upper) for the varibles where **lb is the vector containing the lower bounds** while **ub is the vector containing the upper bounds**. If there is no lower bound or upper bound for a variable it can be written as –inf or inf.

EX4 the following lower bounds and upper bounds

$$\begin{cases} 4 \ge x_1 \\ x_3 \ge 7 \\ -10 \le x_2 \le 8 \end{cases}$$

Must be written in the standard form

$$\Rightarrow \begin{cases} x_1 \le 4 \\ -10 \le x_2 \le 8 \\ 7 \le x_3 \end{cases}$$

Hence lb and ub are given by

$$\underline{lb} = \begin{pmatrix} -\infty \\ -10 \\ 7 \end{pmatrix}, \underline{ub} = \begin{pmatrix} 4 \\ 8 \\ \infty \end{pmatrix}$$

Following the usual instruction they can be saved in MatLab!

Arguments specifications

NONLINEAR CONSTRAINTS (inequality)

$$\underline{c}(\underline{x}) \leq \underline{0}$$

Are the nonlinear inequality constraints **c(x) is the vector containing the expression of the function** that must be not positive.

EX5 the following nonlinear constraints

$$\begin{cases} x^2 - xy \geq -y^4 + 2 \\ xy \leq -x \end{cases}$$

Must be written in the standard form

$$\Rightarrow \begin{cases} -y^4 + 2 - x^2 + xy \leq 0 \\ xy + x \leq 0 \end{cases}$$

Hence c is given by

$$\underline{c}(\underline{x}) = \begin{pmatrix} -y^4 + 2 - x^2 + xy \\ xy + x \end{pmatrix}$$

How to save it in MatLab?

Arguments specifications

NONLINEAR CONSTRAINTS (equality)

$$\underline{ceq}(\underline{x}) = \underline{0}$$

Are the nonlinear equality constraints **ceq(x) is the vector containing the expressions of the function** that must be equal to zero.

EX6 the following nonlinear constraints

$$\begin{cases} xy = z + 2 \\ e^x - z^2 = 4 \end{cases}$$

Must be written in the standard form

$$\Rightarrow \begin{cases} xy - z - 2 = 0 \\ e^x - z^2 - 4 = 0 \end{cases}$$

Hence ceq is given by

$$\underline{ceq}(\underline{x}) = \begin{pmatrix} xy - z - 2 \\ e^x - z^2 - 4 \end{pmatrix}$$

How to save it in MatLab?

How to save it in MatLab the NONLINEAR CONSTRAINTS

In the current directory an m-file must be saved.
For instance, we use the following file namely **nlc_eng.m** (non linear constraints english). The script are the following (with an example):

```
% with the following function non linear constraints can be defined
% the name of the function is nlc_eng (non linear constraints_english)
% the function (m-file) must be saved in the current directory


function [c,ceq]=nlc_eng(x)

% c is the expression of the nonlinear inequality constraints
% with more then 1 constraints create a new row with ;
% if the constraint is not present write c=[]

%%c=[];

c=[x(1)^2+x(2)^2-4;2*x(1)-x(2)];

% ceq is the expression of the nonlinear equality constraints
% with more then 1 constraint create a new row with ;
% if the constraint is not present write ceq=[]


ceq=[];
```

How to save it in MatLab the NONLINEAR CONSTRAINTS

The nonlinear constraints **must be saved in the file nlc_eng**:

- the input **c=[…]** is a column vectors whose elements are the functions of the **inequality constraints** while
- the input **ceq=[…]** is a column vector whose elements are the nonlinear function related to the **equality constraints.**

Use the notation x(1),x(2),…,x(n) for the variables and do not use the punctual operators!

Then save the file with the given constraints!

How to save in MatLab the NONLINEAR CONSTRAINTS

EX7 consider the following nonlinear constraints

$$\begin{cases} x^2 - xy \geq 0 \\ e^x + 2y = 4 \\ 2x^3 + \sqrt{y} \leq 2 \\ y - \ln x = 3 \end{cases}$$

Must be written in the standard form: we consider firstly the inequality constraints and lastly the equality constraints.

$$\Rightarrow \begin{cases} -x^2 + xy \leq 0 \\ 2x^3 + \sqrt{y} - 2 \leq 0 \\ e^x + 2y - 4 = 0 \\ y - \ln x - 3 = 0 \end{cases} \Rightarrow \underline{c} = \begin{pmatrix} -x^2 + xy \\ 2x^3 + \sqrt{y} - 2 \end{pmatrix}, \underline{ceq} = \begin{pmatrix} e^x + 2y - 4 \\ y - \ln x - 3 \end{pmatrix}$$

```
c=[-x(1)^2+x(1)*x(2);2*x(1)^3+sqrt(x(2))-2];
ceq=[exp(x(1))+2*x(2)-4;x(2)-log(x(1))-3];
```

How to solve a constrained otimization problem by using the OPTIMIZATION TOOLBOX

1. Open **optimtool** from
the command window.
2. Select the solver **fmincon**
3. Select the **algoritm** (they use
Different iterative schemes, do not use the trust-region, use one of the other methods, for instance the Active set)
4. Give the **objective function** in the usual manner (it is a mandatory field) **that must be minimized!**
5. Indicate a **start point** in the usual manner (it is a mandatory field). Try with a point, and then change it, verify if the same solution is obtained!
6. Define the **contraints** in the section (they can appear or one or more fields can be empty) as previously explained.
7. If you have **nonlinear constraints**, save them in the file nlc_eng and the write **@nlc_eng** in the correspondent window.

Start!

EX8 consider the following problem

$$\max xy \quad \text{s.t.} \quad \begin{cases} x+2y-10=0 \\ x \geq 1 \\ y \geq 0 \end{cases}$$

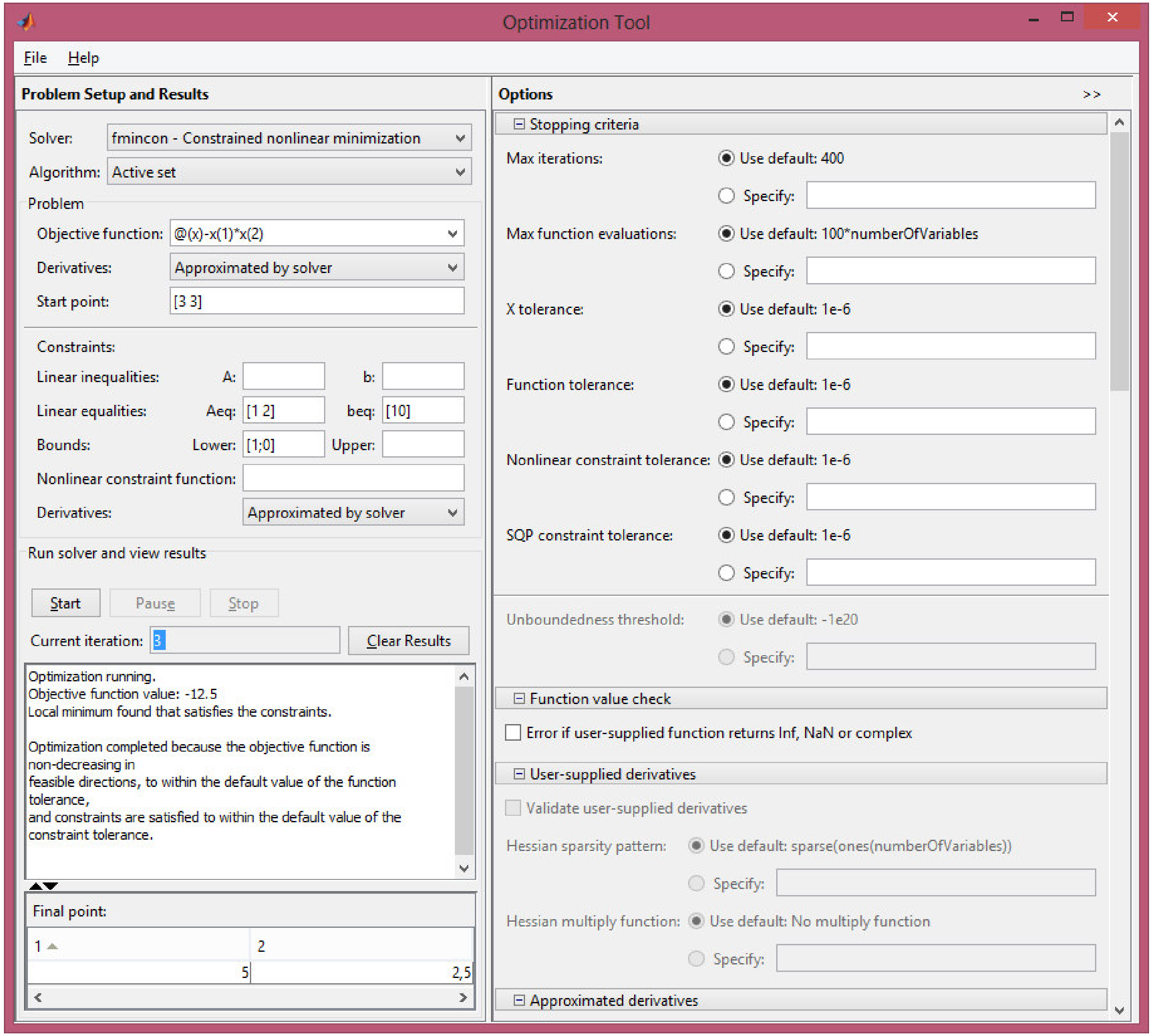
The problem can be re-written in the standard MatLab form:

$$\min -xy \quad \text{s.t.} \quad \begin{cases} x+2y=10 \\ 1 \leq x \\ 0 \leq y \end{cases}$$

With **one linear equality constraint** where **Aeq=[1 2]** and **beq=[10]** and the **lower bound** with **lb=[1;0]**

Using the otimization tool with an arbitrary starting point [3,3] we obtain the following

…EX8



The constrained maximum point is

(5,2.5)

And the objective function values is

12.5

EX9 consider the following problem

$$\min x + 2y + 3z \quad \text{s.t.} \quad \begin{cases} x - y^2 - 1 \le 0 \\ x + zy \ge 1 \end{cases}$$

The problem must be re-written in the standard MatLab form:

$$\min x + 2y + 3z \quad \text{s.t.} \quad \begin{cases} x - y^2 - 1 \le 0 \\ 1 - x - zy \le 0 \end{cases}$$

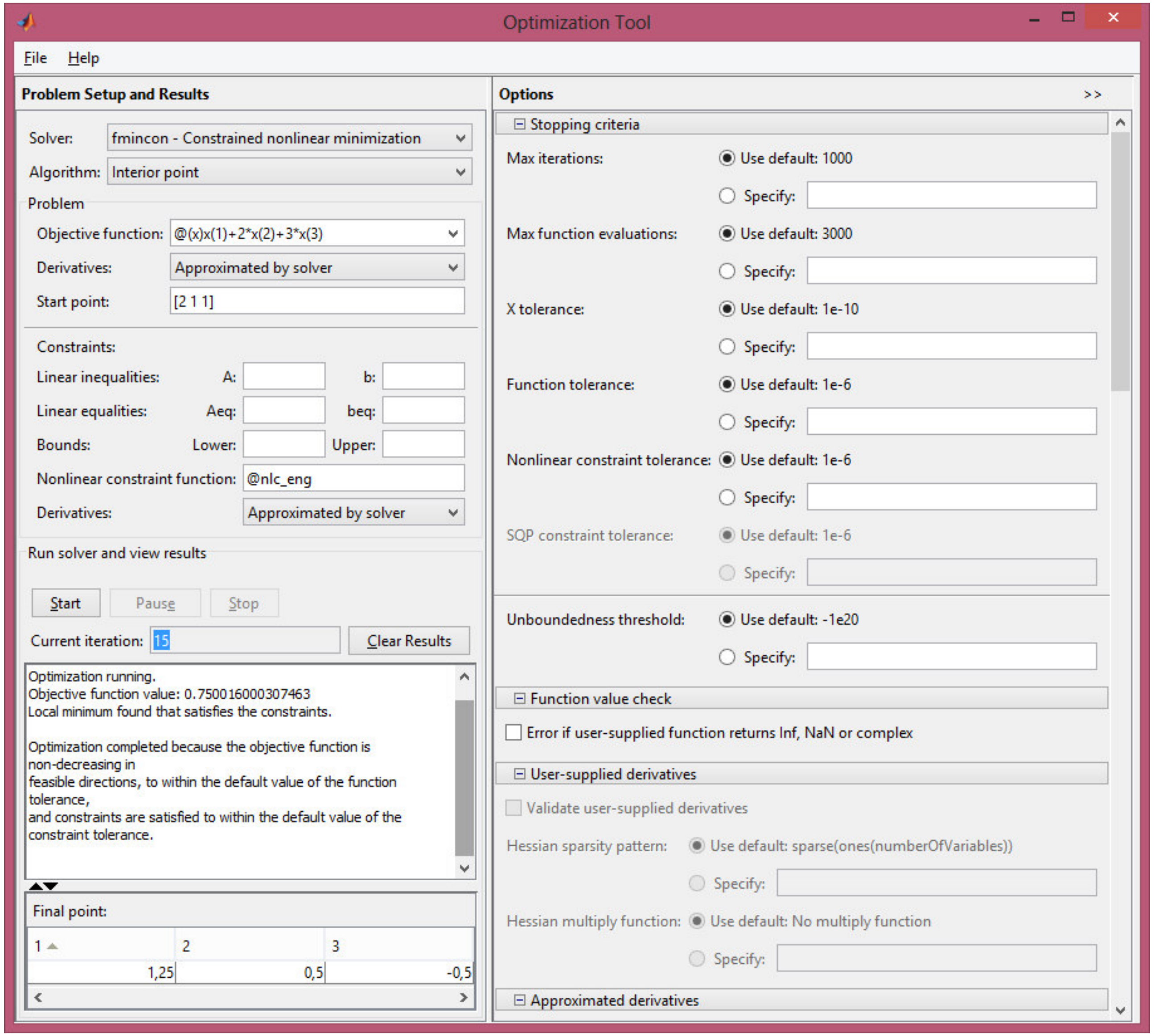
With **two nonlinear inequality constraints**. We have to specify the constraints in the file nlc_eng.m while defining

c=[x(1)-x(2)^2-1;1-x(1)-x(3)*x(2)]

and then save the file in the current directory.

Using the otimization tool with an arbitrary starting point [3,3] we obtain the following.

...EX9



The constrained minimum point is

(1.25,0.5,-0.5)

And the objective function values is

0.75

EX10 Consider 4 risky assets, 1,2,3,4 having respectively the following expected returs: 7%, 3%, 2%, 6%. The protfolio risk depending on the fraction investd in each risky asset is given by $R=3x_1+x_2^2x_3^2+2x_2^2x_4$. The investor can accept a portfolio risk equal AT MOST to 2. Furthermore in the market it is not possible to borrow. Determine how much to invest in each risky asset in order to maximize the expected portfolio return.

The initial problem is formalized as follows:

$$\max 7x_1 + 3x_2 + 2x_3 + 6x_4 \quad \text{s.t.} \quad \begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ 3x_1 + x_2^2 x_3^2 + 2x_2^2 x_4 \le 2 \\ x_i \ge 0, i = 1,2,3,4 \end{cases}$$

The problem must be re-written in the following standard MatLab form:

$$\min -(7x_1 + 3x_2 + 2x_3 + 6x_4) \quad \text{s.t.} \quad \begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ 3x_1 + x_2^2 x_3^2 + 2x_2^2 x_4 - 2 \le 0 \\ x_i \ge 0, i = 1,2,3,4 \end{cases}$$

With **one linear equality constraint**, **one nonlinear inequality constraint** and the **lower bounds**. In the file nlc_eng.m we define:

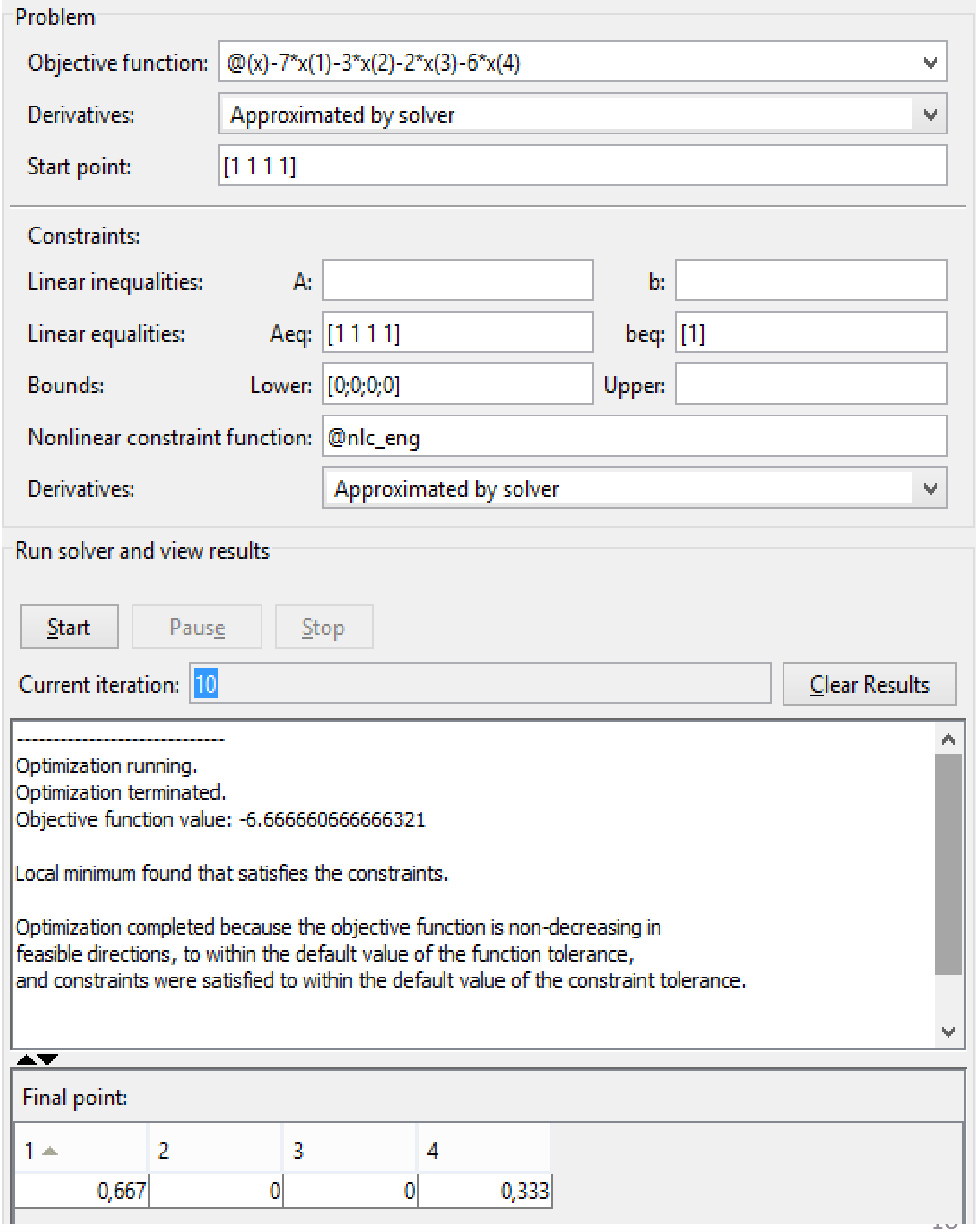
```
c=[3*x(1)+x(2)^2*x(3)^2+2*x(2)^2*x(4)-2];
```

...EX10

By investing a fraction
0.667 in the asset 1
and a fraction
0.333 in the asset 4

the **portfolio return will be equal to 6.667%**

and the portfolio risk will be at the minimum accepted level.



EX11 Consider a firm and 3 inputs (X,Y,Z) in the production process. Consider the following production function depending on the quantities used of each input: f(x,y,z)=xy+yz. The unitary cost of each input is respectively 3, 10, 80. For the production at least two units of the third input must be used. The firm wants to maximize the production by having at most a total cost equal to 300.

The initial problem is formalized as follows:

$$\max xy + yz \quad \text{s.t.} \quad \begin{cases} 3x + 10y + 80z \le 300 \\ x \ge 0, y \ge 0, z \ge 2 \end{cases}$$

The problem must be re-written in the following standard MatLab form:

$$\min -(xy + yz) \quad \text{s.t.} \quad \begin{cases} 3x + 10y + 80z \le 300 \\ x \ge 0, y \ge 0, z \ge 2 \end{cases}$$

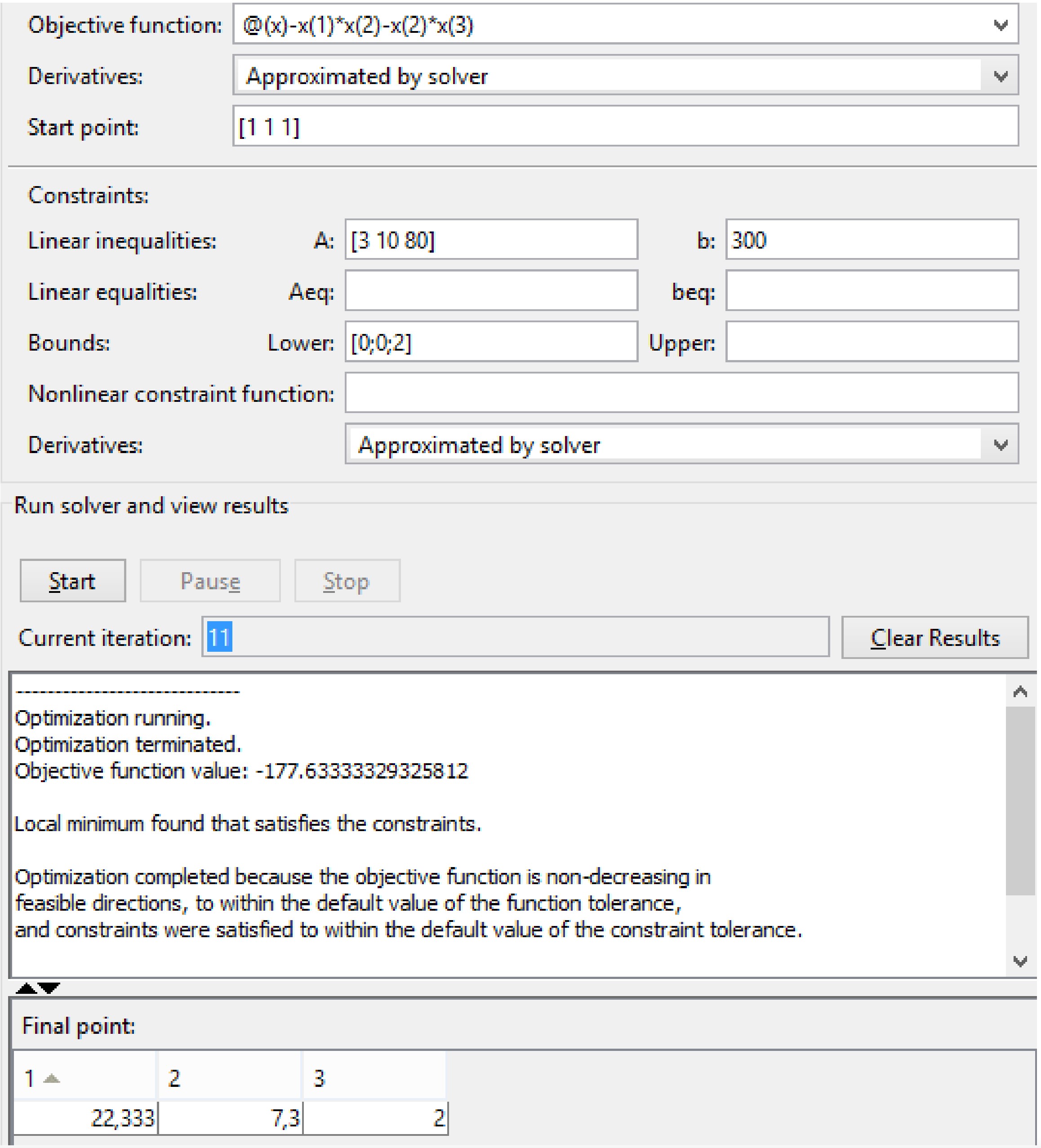
With **one linear inequality constraint**, **lower bounds** (notice that the lower bound for variable z in 2).

We do not need to define any functions in file nlc_eng!

...EX11

The firm must use **22.33 units of input X**, **7.3 units of input Y** and **2 unit of input Z**.

The final quantity that will be produced is **177.63**



HOMEWORKS

EX 1.1

Solve with MatLab the following constrained problems (give the formalized MatLan standard form).

$$1)\max x^2+3y^2 \quad \text{s.t.}\begin{cases}2x+6y-30\le 0\\ x\ge 0, y\ge 0\end{cases}$$

$$2)\min -xyz \quad \text{s.t.}\{0\le x+2y+2z\le 72$$

$$3)\min x_1^2+x_2^2x_3+4x_4^3 \quad \text{s.t.}\begin{cases}x_4-x_2x_3^2-30\le 0\\ x_1=x_2^2\\ x_1\ge 0, x_2\ge 0, x_3\ge -4, x_4\ge 2\end{cases}$$

HOMEWORKS

EX 1.2

Consider a consumer and 4 commodities (X,Y,Z,W) in the market. Consider the following utility function depending on the quantities consumed of each good: f(x,y,z,w)=xy+yzw. The unitary cost of each good is 3, 10, 7, 6 respectively. The consumer wants to consume at list 3 units of the good Y and at least 1 unit of the good W by spending at most 150 euros. Formalize the utility maximization problem under the opportune constraints and solve it with MatLab.

EX 1.3

To produce, a firm needs to use three inputs (1,2,3). The unitary cost of each input is equal to 3,5,9 respectively. To produce output the production function is described by the following formula $Y=3x_1^{0.2}x_2^{0.5}x_3^{0.3}$. Consider that the firm wants to produce 50 units of goods and that the quantity used of the first input must be equal to the quantity used of the second input. Determine by using MatLab how much to use of each input in order to minimize the total cost under the opportune constraints.