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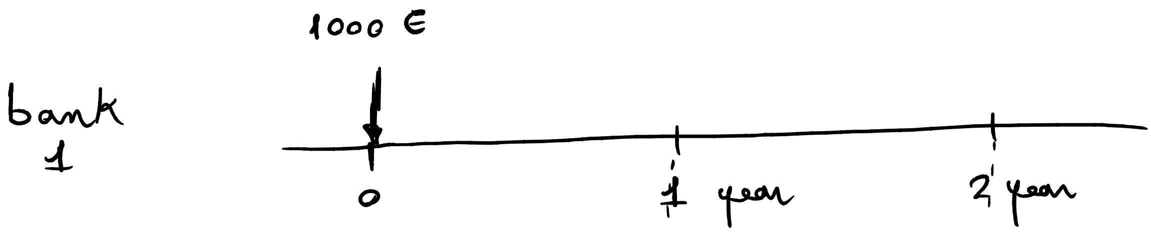
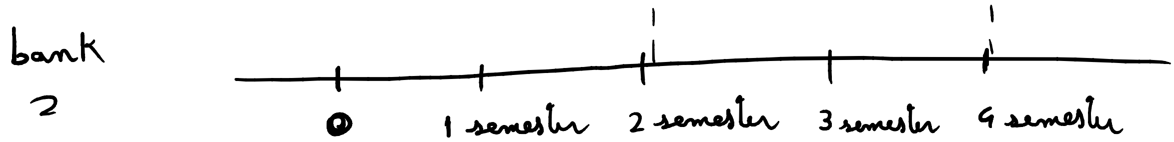
i : interest rate (refers in years)



$$\underbrace{W(1)}_{} = \overset{1000\ €}{\underline{W(0)}}(1 + i)$$

$$W(2) = W(0)(1 + 2i)$$



$$\underset{1 \text{ semester}}{W(\underline{1})} = W(0)(1 + i_2)$$

at bank 2
$W(1)$ is the amount of money after 1 semester

interest rate expressed in semesters

After the second semester:

$$W(2) = W(0)(1 + 2 i_2)$$

After the fourth semester:

$$W(4) = W(0)(1 + 4 i_2)$$

The two amounts of money are equivalent if

$$\underbrace{W(0)\,(1+2i)}_{\text{bank 1}} = \underbrace{W(0)\,(1+4\,i_2)}_{\text{bank 2}}$$

$$\frac{\cancel{W(0)}\,(1+2i)}{\cancel{W(0)}} = \frac{\cancel{W(0)}\,(1+4\,i_2)}{\cancel{W(0)}}$$

$$\cancel{1} + 2i = \cancel{1} + 4\,i_2$$

$$\frac{\cancel{2}\,i}{\cancel{2}} = \frac{\cancel{4}\,i_2}{\cancel{2}}$$

$$\boxed{i = 2\,i_2}$$

i_m : interest rate referring to the m-th part of the year

i_2 : refers to semesters

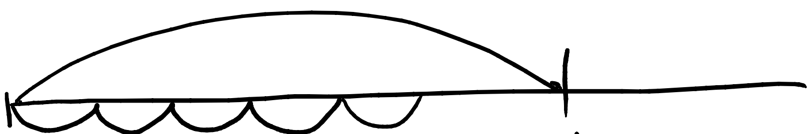
i_3 : refers to every 4 months

i_{12} : refers to months

Generalization

bank 1 (years)

$W(0)\,(1+it)$



t years

bank 2 (m-th parts of the year)

$W(0)\,(1+i_m\,mt)$

$t \cdot m$ parts of the year

(m - th parts of the year)

$t \cdot m$ parts of the year

equivalence :

$$\cancel{W(0)}(1+it) = \cancel{W(0)}(1+i_m \cdot mt)$$

$$\cancel{1} + it = \cancel{1} + i_m \cdot mt$$

$$i\cancel{t} = i_m \cdot m\cancel{t}$$

$$i = i_m \cdot m$$

$$\boxed{i_m = \frac{i}{m}}$$

Bank 3 :

n - th parts of the year

$$i = i_n \cdot n$$

So $\boxed{i_m \cdot m = i_n \cdot n}$

EX5: We have to invest 5000 euro for 15 months and the interest rate referred to a period of 6-months (semi-annual) is i2=0.05.
The final value is given by

1) convert into months

$$i_{12} \cdot 12 = i_2 \cdot 2$$

$$i_{12} = i_2 \cdot \frac{\cancel{2}^{1}}{\cancel{12}_{6}} = \frac{0.05}{6}$$

$$W(15) = 5000\,€\,(1 + i_{12} \cdot 15) =$$

$$= 5000\,€\left(1 + \frac{0.05}{6} \cdot 15\right) = 5625\,€$$

or

2) convert into years

$$i_2 \cdot 2 = i$$

$$i = 2 \cdot 0.05$$

$$15 \text{ months} = \frac{15}{12} \text{ years}$$

$$W(15 \text{ months}) = 5000\,€\left(1 + 2 \cdot 0.05 \cdot \frac{15}{12}\right)$$

3) convert into semesters

$$15 \text{ months} = \frac{15}{6} \text{ semesters}$$

$$W(15 \text{ months}) = 5000\,€\left(1 + 0.05 \cdot \frac{15}{6}\right)$$

$$\underset{\substack{\text{future}\\\text{value}}}{W(t)} = \underset{\substack{\text{principal}\\\text{or present value}}}{W(0)}(1 + it)$$

$$W(0) = \frac{1}{1 + it} W(t)$$

If at time t, I collect $W(t)$ (future value), its value today (present value) is $\frac{1}{1+it} w(t) < w(t)$ so it is discounted. For this reason, the quantity $\frac{1}{1+it}$ is called the discount factor.

RULE OF COMPOUNDING INTEREST (OR EXPONENTIAL INTEREST)

$w(0)$: principal

Bank 1

(years)

$$w(1) = w(0) + i\, w(0) = w(0)(1+i)$$

$$w(2) = w(1) + i\, w(1) =$$

$$= w(1)(1+i) =$$

$$= w(0)(1+i)(1+i) = w(0)(1+i)^2$$

$$w(3) = w(2) + i\, w(2) = w(2)(1+i) =$$

$$= w(0)(1+i)^2 \cdot (1+i) = w(0)(1+i)^3$$

$$w(4) = w(0)(1+i)^4$$

If you wait for n years, you will get:

$$w(n) = w(0)(1+i)^n, \qquad n \in \mathbb{N}$$

More in general:

$$\boxed{w(t) = w(0)(1+i)^t, \qquad t \in \mathbb{R}}$$

$$w(t) = w(0)(1+i)^t \,, \qquad t \in \mathbb{R}$$

EX7: Let i=10%. Then with the exponential rule 1000 euro today is equivalent, after 8 months, to

$$8 \text{ months} = \frac{8}{12} \text{ years}$$

$$w(8 \text{ months}) = 1000\,€\,(1+0.1)^{\frac{8}{12}} = 1065.6\ €$$

```
>> 1000*(1 + .1)^(8/12)

ans =

   1.0656e+03
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$$1.0656 \cdot 10^3 = 1065.6$$

examples

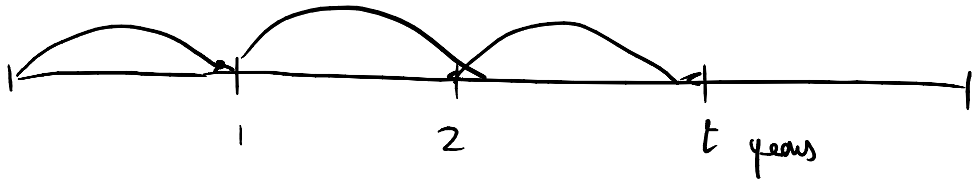
$$50 = 5 \cdot 10^1 = 5e1$$

$$0.01 = 10^{-2} = 1e-2$$

$$r(t) = \text{growth factor} := \frac{w(t)}{w(0)}$$

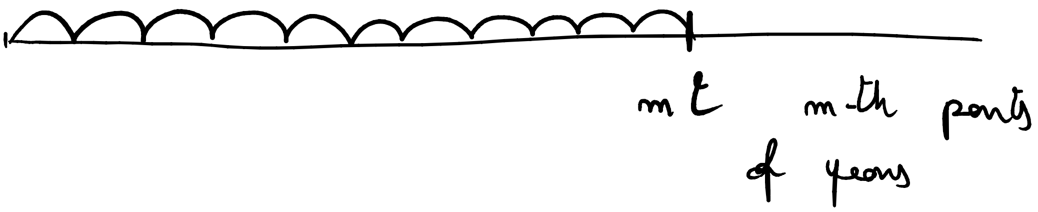
$$r(t) = \frac{w(t)}{w(0)} = \frac{\cancel{w(0)}(1+i)^t}{\cancel{w(0)}} = (1+i)^t$$

bank 1



bank 2

m-th parts of the year



$$\frac{\cancel{W(0)}\,(1+i)^{t}}{\cancel{W(0)}} = \frac{\cancel{W(0)}\,(1+i_m)^{mt}}{\cancel{W(0)}}$$

$$\left[(1+i)^{\cancel{t}}\right]^{\frac{1}{\cancel{t}}} = \left[(1+i_m)^{m\cdot\cancel{t}}\right]^{\frac{1}{\cancel{t}}}$$

$$\left[1+i\right]^{\frac{1}{m}} = \left[(1+i_m)^{\cancel{m}}\right]^{\frac{1}{\cancel{m}}}$$

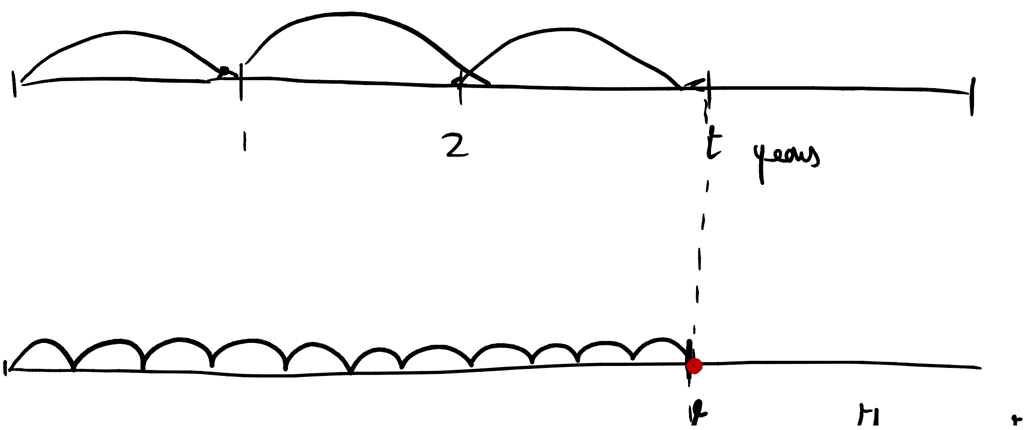
$$(1+i)^{\frac{1}{m}} = 1+i_m$$

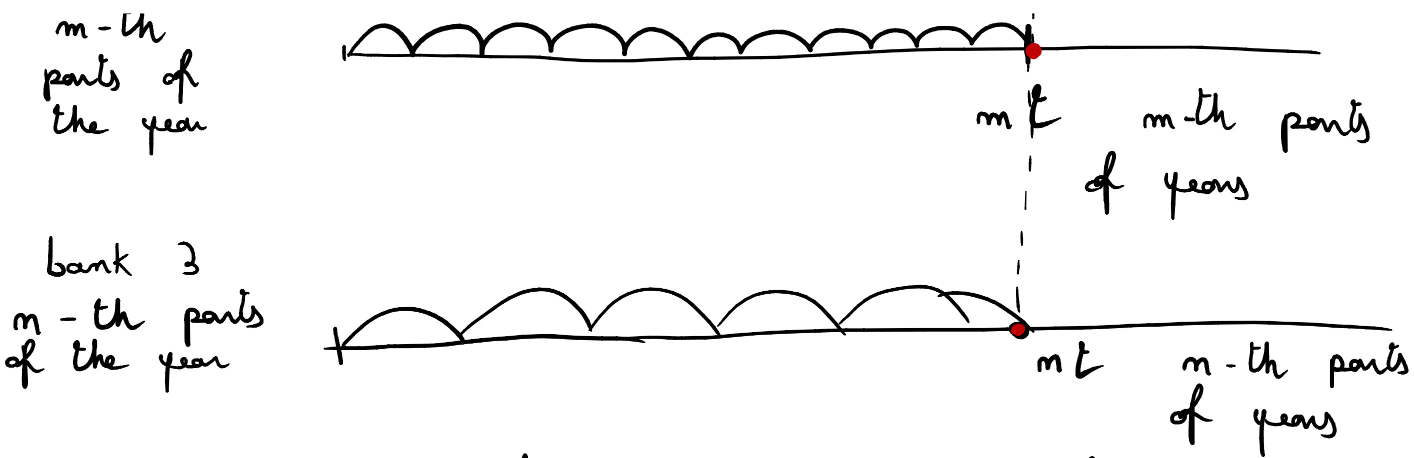
$$i_m = (1+i)^{\frac{1}{m}} - 1$$

bank 1

bank 2

m-th parts of





$$\cancel{W(0)}\,(1+i_m)^{mt} = \cancel{W(0)}\,(1+i_n)^{nt}$$

$$\left[(1+i_m)^{m\cancel{t}}\right]^{\frac{1}{\cancel{t}}} = \left[(1+i_n)^{n\cancel{t}}\right]^{\frac{1}{\cancel{t}}}$$

$$\left[(1+i_m)^{\cancel{m}}\right]^{\frac{1}{\cancel{m}}} = \left[(1+i_n)^{n}\right]^{\frac{1}{m}}$$

$$1+i_m = (1+i_n)^{\frac{n}{m}}$$

$$\boxed{i_m = (1+i_n)^{\frac{n}{m}} - 1}$$

If n is in years, n = 1 you get, as before:

$$i_m = (1+i)^{\frac{1}{m}} - 1$$

EX7: Let i=10%. Then with the exponential rule 1000 euro today is equivalent, after 8 months, to

8 2

$$i_{12} = (1+i)^{\frac{1}{12}} - 1 = (1+0.1)^{\frac{1}{12}} - 1 =$$

$$i_{12} = (1+i) - 1 = (1+0.1) - 1 =$$

$$= 1.1^{1/12} - 1 = 0.008$$

$$W(8 \text{ months}) = W(0)\,(1+i_{12})^{8} =$$

$$= 1000\,€\,(1.008)^{8} = 1065.8\,€$$