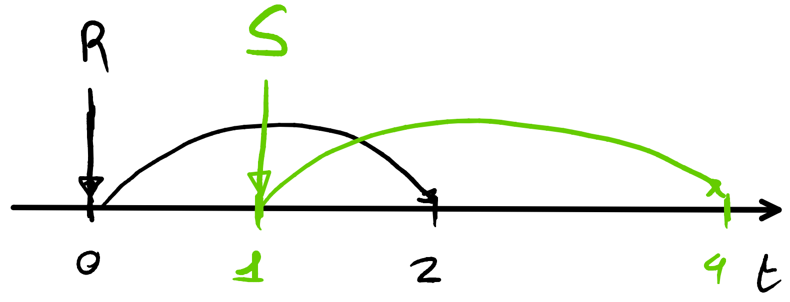
giovedì 29 febbraio 2024 09:00

Compound interest law:

Until now the interest rate has always been assumed to be fixed. i



$$w(0) = R$$

$$w(2) = R(1+i)^2$$

$$w(4) = w(1)(1+i)^3 =$$
$$= S(1+i)^3$$

Now we tackle the problem where the interest rate is no longer fixed

1) spot interest rate $i(0, t)$

The interest rate depends on the duration of your investment

2) forward interest rate $i(0, t-1, t)$

The generalization is $i(0, t_1, t_2)$ with $t_2 \geq t_1$

It is the interest rate if today (i.e. time 0) you decide to make an investment at time t_1 until time t_2.

EX26: consider a zero coupon bond (ZCB) that is quoted 94.3, ending in 6 months, nominal value (rembursmente value) 100.

6 monts = 0.5 years

$$W(t) = W(0)\,(1 + i(0,t))^{t}$$

$$W(0.5) = W(0)\,(1 + \underbrace{i(0,\tfrac{1}{2})}_{?})^{\frac{1}{2}}$$

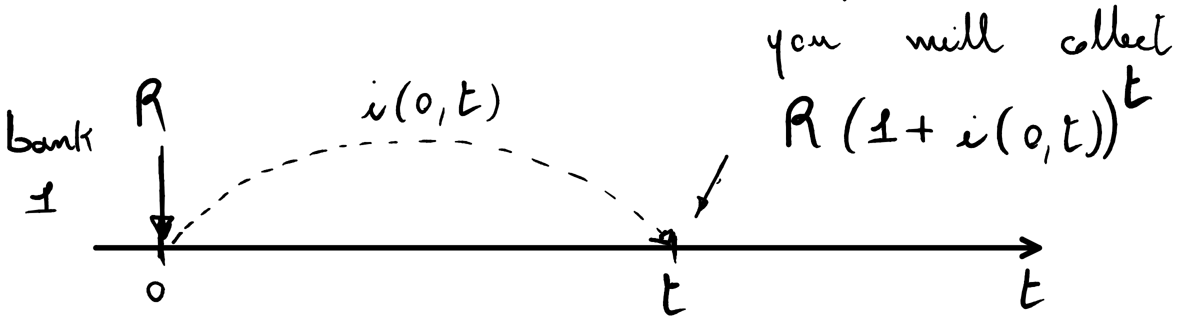
$$100 = 94.3\,(1 + i(0,\tfrac{1}{2}))^{\frac{1}{2}}$$

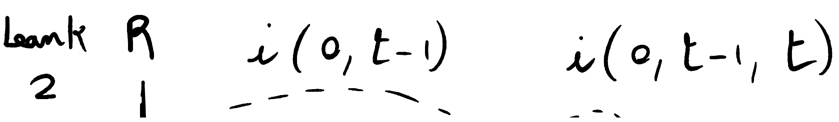
$$\frac{100}{94.3} = \sqrt{1 + i(0,\tfrac{1}{2})}$$

$$\left(\frac{100}{94.3}\right)^2 = 1 + i(0,\tfrac{1}{2})$$

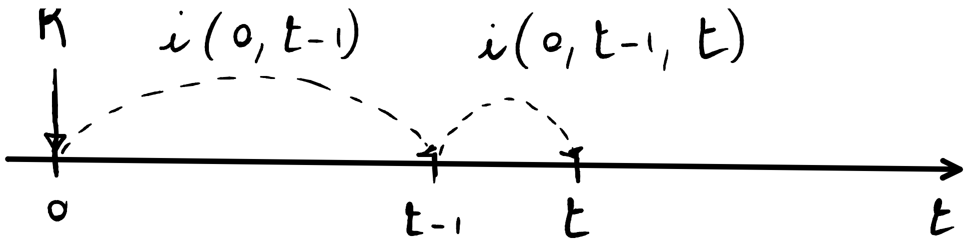
$$i(0,\tfrac{1}{2}) = \left(\frac{100}{94.3}\right)^2 - 1 = 0.1245$$

The spot interest rate and the forward interest rate are not independent but they relate to each other according to the following relationship:





Bank 2



At time $t-1$ you collect:

$$W(t-1) = R(1+i(0,t-1))^{t-1}$$

Then you reinvest it and at time t you have:

$$W(t) = W(t-1)(1+i(0,t-1,t))^{1} =$$

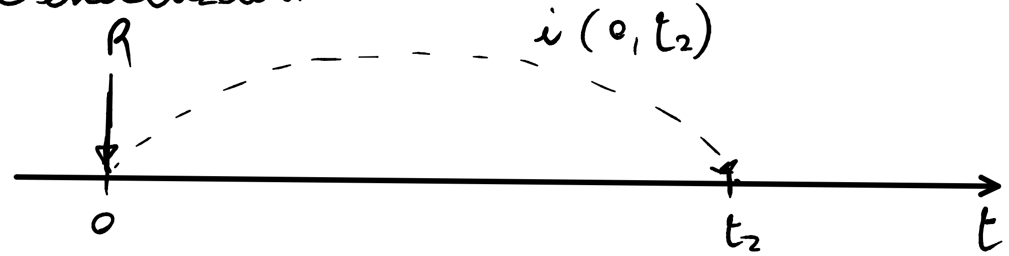
$$= R(1+i(0,t-1))^{t-1}(1+i(0,t-1,t))$$

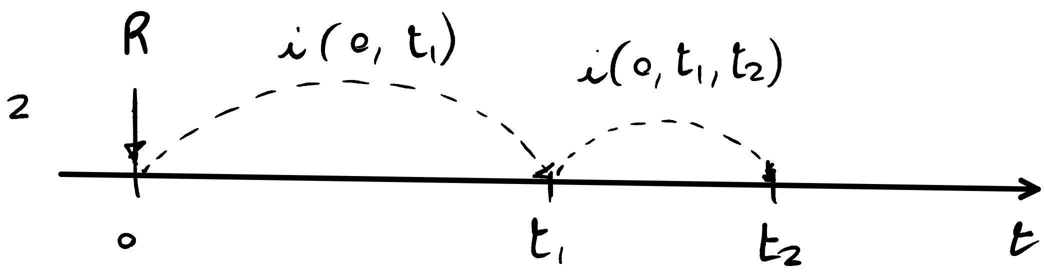
$$\frac{\cancel{R}(1+i(0,t))^{t}}{(1+i(0,t-1))^{t-1}} = \frac{\cancel{R}\,\cancel{(1+i(0,t-1))^{t-1}}\,\underline{(1+i(0,t-1,t))}}{\cancel{(1+i(0,t-1))^{t-1}}}$$

$$\boxed{i(0,t-1,t) = \frac{(1+i(0,t))^{t}}{(1+i(0,t-1))^{t-1}} - 1}$$

Generalization:

1





① $w(t_2) = R\left(1+i(0,t_2)\right)^{t_2}$

② $w(t_2) = R\left(1+i(0,t_1)\right)^{t_1}\left(1+i(0,t_1,t_2)\right)^{t_2-t_1}$

$$\frac{\cancel{R}\left(1+i(0,t_2)\right)^{t_2}}{\left(1+i(0,t_1)\right)^{t_1}} = \frac{\cancel{R}\,\cancel{\left(1+i(0,t_1)\right)^{t_1}}\left(1+i(0,t_1,t_2)\right)^{t_2-t_1}}{\cancel{\left(1+i(0,t_1)\right)^{t_1}}}$$

$$\left[\left(1+i(0,t_1,t_2)\right)^{\cancel{t_2-t_1}}\right]^{\frac{1}{\cancel{t_2-t_1}}} = \left[\frac{\left(1+i(0,t_2)\right)^{t_2}}{\left(1+i(0,t_1)\right)^{t_1}}\right]^{\frac{1}{t_2-t_1}}$$

$$\boxed{i(0,t_1,t_2) = \left[\frac{\left(1+i(0,t_2)\right)^{t_2}}{\left(1+i(0,t_1)\right)^{t_1}}\right]^{\frac{1}{t_2-t_1}} - 1}$$

EX29: Suppose the spot rates of interest for investment horizons of 1, 2, 3 and 4 years are, respectively, 4%, 4.5%, 4.5%, and 5%. Calculate the forward rates of interest for t = 1, 2, 3 and 4.

$i(0,1,2) = ?$ $\qquad i(0,2,3) = ?$

$i(0,3,4) = ?$

$$i(0,t-1,t) = \frac{\left(1+i(0,t)\right)^{t}}{\quad} - 1$$

$$i(0, t-1, t) = \frac{(1 + i(0,t))^t}{(1 + i(0,t-1))^{t-1}} - 1$$

$$i(0,1,2) = \frac{(1 + i(0,2))^2}{(1 + i(0,1))^1} - 1 =$$

$$= \frac{(1 + 0.045)^2}{1 + 0.04} - 1 = 0.05 = 5\%$$

$$i(0,2,3) = \frac{(1 + i(0,3))^3}{(1 + i(0,2))^2} - 1 =$$

$$= \frac{(1 + 0.045)^3}{(1 + 0.045)^2} - 1 =$$

$$= 1 + 0.045 - 1 = 0.045 = 4.5\%$$

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>> (1 + 0.045)^2/(1 + 0.04) - 1

ans =

    0.0500
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SIMPLE AVERAGE

N numbers : $x_1, x_2, \ldots, x_N$

N numbers : $x_1, x_2, \dots, x_N$

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_N}{N} =$$

$$= \frac{1}{N} x_1 + \frac{1}{N} x_2 + \dots + \frac{1}{N} x_N$$

Weighted average

N numbers : $x_1, x_2, \dots, x_N$

N weights : $w_1, w_2, \dots, w_N$

Their sum is denoted by $W = w_1 + w_2 + \dots + w_N$

$$\bar{x}_w = \frac{w_1 x_1 + w_2 x_2 + \dots + w_N x_N}{W}$$

DURATION:

$$OF = \{(x_1, x_2, \dots, x_m); (t_1, t_2, \dots, t_m)\}$$

We assume $x_k \geq 0 \quad \forall \; k \in \{1, \dots, m\}$

for all indexes

The duration is the weighted average of the time of the cash flow where the weights are the present values of the cash flows taking into account the term structure of the spot rates of interests.

A term structure is the list of the spot rates of interests, i.e., $i(0, t_1)$, $i(0, t_2)$, ..., $i(0, t_m)$.

At time t_1 you invest x_1.
Its present value is:

$$\underbrace{x_1 (1 + i(0, t_1))^{-t_1}}_{w_1}$$

At time t_k you invest x_k. Its present value is:

$$\underbrace{x_k (1 + i(0, t_k))^{-t_k}}_{w_k}$$

$$D = \frac{x_1\left(1+i(0,t_1)\right)^{-t_1}\cdot t_1 + x_2\left(1+i(0,t_2)\right)^{-t_2} t_2 + \dots + x_m\left(1+i(0,t_m)\right)^{-t_m} t_m}{x_1\left(1+i(0,t_1)\right)^{-t_1} + x_2\left(1+i(0,t_2)\right)^{-t_2} + \dots + x_m\left(1+i(0,t_m)\right)^{-t_m}}$$

If the interest rate is assumed to be constant, i.e. $i(0,t_k) = i$ for all k, we get the so-called Macauly duration:

$$D_{MAC} = \frac{x_1(1+i)^{-t_1} t_1 + x_2(1+i)^{-t_2} t_2 + \dots + x_m(1+i)^{-t_m} t_m}{x_1(1+i)^{-t_1} + x_2(1+i)^{-t_2} + \dots + x_m(1+i)^{-t_m}}$$