$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \quad \text{or} \quad \underline{\underline{A}} = [a_{ij}]$$

i refers to the rows

j refers to the columns

$$\underline{\underline{A}} = \begin{array}{c} \begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \\ \begin{array}{ccc} a_{21} & a_{22} & a_{23} \end{array} \end{array}$$

is a matrix with 2 rows and 3 columns

$$\underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix}$$

Addition between 2 matrices: it is possible provided that the two matrices have the same sizes.

$$\underline{\underline{A}} + \underline{\underline{B}} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mm}+b_{mm} \end{bmatrix}$$

Multiplication by a scalar k

$$k\underline{\underline{A}} = \begin{bmatrix} k a_{11} & k a_{12} & \cdots & k a_{1m} \\ k a_{21} & k a_{22} & \cdots & k a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ k a_{m1} & k a_{m2} & \cdots & k a_{mm} \end{bmatrix}$$

EX:
$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \qquad \underline{\underline{B}} = \begin{bmatrix} 5 & 0 \\ -6 & 7 \end{bmatrix}$$

$$5\,\underline{\underline{A}} - 2\,\underline{\underline{B}} = 5 \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} - 2 \begin{bmatrix} 5 & 0 \\ -6 & 7 \end{bmatrix} =$$

$$= \begin{bmatrix} 5 \cdot 1 & 5 \cdot 2 \\ 5 \cdot 3 & 5 \cdot (-4) \end{bmatrix} + \begin{bmatrix} -2 \cdot 5 & -2 \cdot 0 \\ -2 \cdot (-6) & -2 \cdot 7 \end{bmatrix} =$$

$$= \begin{bmatrix} 5 & 10 \\ 15 & -20 \end{bmatrix} + \begin{bmatrix} -10 & 0 \\ 12 & -14 \end{bmatrix} =$$

$$= \begin{bmatrix} 5 + (-10) & 10 + 0 \\ 15 + 12 & -20 + (-14) \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ 27 & -34 \end{bmatrix}$$

row matrix (row vector)

$$\underline{\underline{A}} = [a_1 \quad a_2 \quad \dots \quad a_m]$$

column matrix (or column vector)

$$\underline{\underline{B}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Provided they have the same number of elements, we can define their multiplication as:

$$\underline{\underline{A}}\ \underline{\underline{B}} = [a_1 \quad a_2 \dots a_m] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_m b_m =$$

$$\underline{\underline{A}}\ \underline{\underline{b}} = [a_1 \ \ a_2 \ \dots \ a_m] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \underbrace{a_1 b_1 + a_2 b_2 + \dots + a_m b_m}_{} =$$

$$= \sum_{k=1}^{m} a_k b_k$$

EX

$$[7 \ \ -4 \ \ \ 5] \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = 7 \cdot 3 - 4 \cdot 2 + 5 \cdot (-1) =$$

$$= 21 - 8 - 5 = 8$$

MULTIPLICATION BETWEEN TWO MATRICES

$\underline{\underline{A}}\ \underline{\underline{B}}$ is possible provided that:

the number of colums of $\underline{\underline{A}}$ is equal to the number of rows of $\underline{\underline{B}}$.

$\underline{\underline{A}}$: $\underline{m \times p}$ $\quad$ m rows, p columns

$\underline{\underline{B}}$: $p \times n$

$\underline{\underline{C}} = \underline{\underline{A}}\ \underline{\underline{B}}$ $\quad$ $m \times n$

$$\overset{\underline{\underline{A}}}{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ip} \end{bmatrix}} \overset{\underline{\underline{B}}}{\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \end{bmatrix}} = \overset{\underline{\underline{C}}}{\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & \boxed{c} & \vdots \end{bmatrix}}$$

$$
\begin{bmatrix} \vdots & \vdots & \ddots & \vdots \\ \boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{ip}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix}
\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ b_{p1} & b_{p2} & \cdots \; b_{pj} \; \cdots & b_{pm} \end{bmatrix}
=
\begin{bmatrix} \vdots & \boxed{c_{ij}} & \vdots \\ c_{m1} & c_{m2} \; \cdots & c_{mm} \end{bmatrix}
$$

i-th row

j-th column

The element of the resulting matrix $\underline{\underline{C}}$ in position (i, j) i.e. c_{ij} is given by the multiplication of the i-th row of the matrix $\underline{\underline{A}}$ by the j-th column of the matrix $\underline{\underline{B}}$.

$$
c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ip} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix} = \sum_{k=1}^{p} a_{ik} b_{kj}
$$

$$
\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \qquad \underline{\underline{B}} = \begin{bmatrix} 5 & 0 \\ -6 & 7 \end{bmatrix}
$$

$$
\underline{\underline{A}}\,\underline{\underline{B}} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ -6 & 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot (-6) & 1 \cdot 0 + 2 \cdot 7 \\ 3 \cdot 5 - 4 \cdot (-6) & 3 \cdot 0 - 4 \cdot 7 \end{bmatrix} =
$$

$$
= \begin{bmatrix} 5 - 12 & 14 \\ 15 + 24 & -28 \end{bmatrix} = \begin{bmatrix} -7 & 14 \\ 39 & -28 \end{bmatrix}
$$

$$
\underline{\underline{B}}\,\underline{\underline{A}} = \begin{bmatrix} 5 & 0 \\ -6 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 + 0 \cdot 3 & 5 \cdot 2 + 0 \cdot (-4) \\ -6 \cdot 1 + 7 \cdot 3 & -6 \cdot 2 + 7 \cdot (-4) \end{bmatrix} =
$$

$$
= \begin{bmatrix} 5 & 10 \\ \cdots & \cdots \end{bmatrix} \neq \underline{\underline{A}}\,\underline{\underline{B}}
$$

$$= \begin{bmatrix} 5 & 10 \\ 15 & -40 \end{bmatrix} \neq \underline{\underline{A}}\,\underline{\underline{B}}$$

The multiplication between two matrices is not commutative.

TRANSPOSE OF A MATRIX

$$\underline{\underline{A}} = [a_{ij}] \qquad \underline{\underline{A}}^T = [a_{ji}]$$

$$\underline{\underline{C}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \underline{\underline{C}}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

* SYMMETRIC MATRICES

A matrix is symmetric if $\underline{\underline{A}}^T = \underline{\underline{A}}$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \underline{\underline{A}}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \neq \underline{\underline{A}} \quad \text{not symmetric}$$

$$\underline{\underline{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad \underline{\underline{B}}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \underline{\underline{B}}^T$$

* SQUARE MATRICES

A matrix is square if its number of rows is equal to its number of column. and this number is said to be the order of the matrix equal to its number of column. and this number is said to be the order of the matrix.

In other word $m = n$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

IDENTITY MATRIX (square)

$$\underline{\underline{I}} \quad \text{or} \quad \underline{\underline{I}}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}} = \underline{\underline{I}}\,\underline{\underline{A}}$$

INVERSE OF A MATRIX

Given the (square) matrix $\underline{\underline{A}}$, the inverse of $\underline{\underline{A}}$ is that (square) matrix $\underline{\underline{B}}$ (if it exists) such that:

$$\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{I}}$$ This matrix is denoted by $\underline{\underline{A}}^{-1}$

Generalization of the inverse of a number

$$a \cdot a^{-1} = a \cdot \frac{1}{a} = 1 = a^{-1} \cdot a$$

Here the requirement is $a \neq 0$

$$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$$

$$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$$

Here the requirement is $\det \underline{\underline{A}} \neq 0$

EX:

$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\underline{\underline{A}}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\underline{\underline{A}}^{-1}\underline{\underline{A}} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & \cancel{bd-bd} \\ \cancel{-ac+ac} & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This operation is possible provided that $ad-bc \neq 0$ and this quantity is just the determinant of $\underline{\underline{A}}$.

SYSTEMS OF EQUATIONS

$$\begin{cases} 3x + 4y + 6z = 3 \\ x + 2y + 3z = 1 \\ 10x + 5y - 3z = -4 \end{cases} \qquad (*)$$

$$\begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 3 \\ 10 & 5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} \qquad \text{matrix form of } (*)$$

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

$$\left[\begin{array}{ccc|c} 3 & 4 & 6 & 3 \\ 1 & 2 & 3 & 1 \\ 10 & 5 & -3 & -4 \end{array}\right]$$

augmented matrix

$$[\underline{\underline{A}} \mid \underline{b}]$$

elementary row operations:

1) interchange two rows of the system

$$\begin{cases} 3x + 4y + 6z = 3 \\ x + 2y + 3z = 1 \\ 10x + 5y - 3z = -4 \end{cases} \qquad \begin{cases} x + 2y + 3z = 1 \\ 3x + 4y + 6z = 3 \\ 10x + 5y - 3z = -4 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & 4 & 6 & 3 \\ 1 & 2 & 3 & 1 \\ 10 & 5 & -3 & -4 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 4 & 6 & 3 \\ 10 & 5 & -3 & -4 \end{array}\right]$$

tilde

"it is equivalent to"

2) change a row by adding to it a multiple of another row

3) multiply each element in a by the same non-zero element

$$x + 2y + 3z = 1$$
$$3x + 4y + 6z = 3$$

$$3x + 4y + 6z - 3(x + 2y + 3z) = 3 - 3(1)$$

$$\cancel{3x} + 4y + 6z - \cancel{3x} - 6y - 9z = 0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 4 & 6 & 3 \\ 10 & 5 & -3 & -4 \end{array}\right] \begin{array}{l} R_2 \leftarrow R_2 - 3R_1 \\ \sim \\ R_3 \leftarrow R_3 - 10R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -2 & -3 & 0 \\ 0 & -15 & -33 & -14 \end{array}\right] \quad R_2 \leftarrow -\frac{1}{2} R_2 \quad \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & -15 & -33 & -14 \end{array}\right] \begin{array}{l} \sim \\ R_3 \leftarrow R_3 + 15R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{21}{2} & -14 \end{array}\right]$$

echelon form

pivots

$$\begin{cases} x + 2y + 3z = 1 \\ y + \frac{3}{2}z = 0 \\ -\frac{21}{2}z = -14 \end{cases} \quad \cdots \quad \begin{cases} x = 1 \\ y = -2 \\ z = \frac{4}{3} \end{cases}$$

Gauss elimination algorithm.