The rank of a matrix is the number of pivots of the same matrix reduced to an echelon form.

$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 2x_1 + 2x_2 + x_3 = 0 \\ x_1 + x_2 + 2x_3 = -1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ \enclose{circle}{2} & 2 & 1 & 0 \\ \enclose{circle}{1} & 1 & 2 & -1 \end{array}\right] \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} \enclose{circle}{1} & 1 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 3 & -2 \end{array}\right] R_3 \leftarrow R_3 - R_2 \quad \sim$$

$$\left[\begin{array}{ccc|c} \enclose{circle}{1} & 1 & -1 & 1 \\ 0 & 0 & \enclose{circle}{3} & -2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$

$$\text{rank}\left(\left[\underline{\underline{A}} \mid \underline{b}\right]\right) = 2$$

$$\text{rank}\left(\underline{\underline{A}}\right) = 2$$

$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 3x_3 = -2 \end{cases}$$

$$\cancel{0x_1 + 0x_2 + 0x_3 = 0} \qquad \cancel{0 = 0}$$

$$\begin{cases} x_1 = -x_2 - \frac{2}{3} + 1 \\ x_3 = -\frac{2}{3} \end{cases}$$

$$\begin{cases} x_1 = -x_2 + \frac{1}{3} \\ x_3 = -\frac{2}{3} \end{cases}$$

$$\begin{cases} x_1 = -t + \frac{1}{3} \\ x_2 = t \quad \text{free variable} \qquad t \in \mathbb{R} \\ x_3 = -\frac{2}{3} \end{cases}$$

$$S = \left\{ \left(-t + \frac{1}{3}, t, -\frac{2}{3} \right) \,\middle|\, t \in \mathbb{R} \right\}$$

$$\infty$$

$$\begin{cases} -2x_1 + x_2 + x_3 = 1 \\ x_1 - 2x_2 + x_3 = -2 \\ x_1 + x_2 - 2x_3 = 4 \end{cases}$$

$$\vdots$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} \operatorname{rank}([\underline{\underline{A}} \;\; \underline{b}]) = 3 \\ \operatorname{rank}(\underline{\underline{A}}) = 2 \end{array} \neq$$

$$\begin{cases} x_1 - 2x_2 + x_3 = -2 \\ x_2 - x_3 = 1 \\ 0x_1 + 0x_2 + 0x_3 = 1 \end{cases} \quad \text{i.e.,} \quad 0 = 1$$

the system is impossible

Yesterday:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{21}{2} & -14 \end{array}\right] \qquad \begin{array}{l} \operatorname{rank}(\underline{\underline{A}}) = 3 \\ \operatorname{rank}([\underline{\underline{A}} \,|\, \underline{b}]) = 3 \end{array}$$

$$\underbrace{\qquad\qquad\qquad}_{\underline{\underline{A}}}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{[\underline{\underline{A}} \,|\, \underline{b}]}$$

Rouché - Capelli theorem: a system of linear equations with n variables has a solution if and only if the rank of its coefficient matrix $\underline{\underline{A}}$ is equal to the rank of its augmented matrix $[\underline{\underline{A}} \mid \underline{b}]$. In this case, there are $\infty^{n - \text{rank}(\underline{\underline{A}})}$ solutions (with the convention that $\infty^0 = 1$)

THE DETERMINANT

For any square matrix, we define a number called the determinant with the property that the square matrix is nonsingular (i.e., invertible) iif its determinant is not 0.

$$a \cdot a^{-1} = 1 \qquad\qquad a \neq 0$$

For 1×1 matrices:

$a = [a]$ scalar: 1 row × 1 column

$$\det(a) = a$$

For 2×2 matrices

$$\underline{\underline{A}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{-} \qquad \det \underline{\underline{A}} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

DEFINITION: let $\underline{\underline{A}}$ be an $n \times n$ matrix and let $\underline{\underline{A}}_{ij}$ be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from $\underline{\underline{A}}$. Then, the scalar

$$M_{ij} = \det \underline{\underline{A}}_{ij}$$

is called the (i,j)-th <u>minor</u> of $\underline{\underline{A}}$ and the scalar

$$c_{ij} = (-1)^{i+j} M_{ij}$$ is called

the (i,j)-th cofactor of $\underline{\underline{A}}$.

LAPLACE RULE

The determinant of an $n \times n$ matrix $\underline{\underline{A}}$ can be defined inductively as follows:

- for $n = 1$, $\det \underline{\underline{A}} = \det(a_{11}) = a_{11}$
- for $n > 1$:

$$\det \underline{\underline{A}} = \sum_{j=1}^{n} \underbrace{(-1)^{i+j}} a_{ij} \det \underline{\underline{A}}_{ij}$$

(expression along row i) or

$$\det A \quad \sum^{n} \quad (-1)^{i+j} \quad \det A$$

$$\det \underline{\underline{A}} = \sum_{i=1}^{m} \underbrace{(-1)^{i+j}} a_{ij} \det \underline{\underline{A}}_{ij}$$

(expression along column j)

1) choose any row or any column

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ -1 & 2 & 1 \end{bmatrix}$$

2) build the "chessboard"

$(-1)^{i+j}$

$i=1 \quad j=1 \qquad (-1)^{2} = 1$

$i=1 \quad j=2 \qquad (-1)^{3} = -1$

$$\begin{bmatrix} {}^{+}1 & {}^{-}2 & {}^{+}3 \\ {}^{-}1 & {}^{+}0 & {}^{-}-1 \\ {}^{+}-1 & {}^{-}2 & {}^{+}1 \end{bmatrix}$$

3) compute

$$\det \underline{\underline{A}} = \sum_{j=1}^{\overset{3}{m}} \underbrace{(-1)^{\overset{1}{i}+j}} a_{\overset{1}{i}j} \det \underline{\underline{A}}_{\overset{1}{i}j} =$$

$$= \underbrace{(-1)^{2}} a_{11} \det \underline{\underline{A}}_{11} + \underbrace{(-1)^{3}} a_{12} \det \underline{\underline{A}}_{12} + \underbrace{(-1)^{4}} a_{13} \det \underline{\underline{A}}_{13} =$$

you take them from the chessboard

$$= \textcircled{1}\ \det\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} - \textcircled{2}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \textcircled{3}\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} =$$

$$\begin{bmatrix} {}^{+}\textcircled{1} & {}^{-}2 & {}^{+}3 \\ {}^{-}1 & {}^{+}0 & {}^{-}{-1} \\ {}^{+}{-1} & {}^{-}2 & {}^{+}1 \end{bmatrix} \qquad \begin{bmatrix} {}^{+}1 & {}^{-}\textcircled{2} & {}^{+}3 \\ {}^{-}1 & {}^{+}0 & {}^{-}{-1} \\ {}^{+}{-1} & {}^{-}2 & {}^{+}1 \end{bmatrix} \qquad \begin{bmatrix} {}^{+}1 & {}^{-}2 & {}^{+}\textcircled{3} \\ {}^{-}1 & {}^{+}0 & {}^{-}{-1} \\ {}^{+}{-1} & {}^{-}2 & {}^{+}1 \end{bmatrix}$$

$$= 0 \cdot 1 - 2 \cdot (-1) - 2\big(1 \cdot 1 - (-1)(-1)\big) + 3\big(1 \cdot 2 - (-1) \cdot 0\big) =$$

$$= 2 - \cancel{2 \cdot 0} + 3(2) = 8$$

or

$$\underline{A} = \begin{bmatrix} {}^{+}1 & {}^{-}2 & {}^{+}3 \\ {}^{-}1 & {}^{+}0 & {}^{-}{-1} \\ {}^{+}{-1} & {}^{-}2 & {}^{+}1 \end{bmatrix} = -2\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \cancel{0\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}}$$

$$-2\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} =$$

$$= -2\cancel{\big(1 \cdot 1 - (-1)(-1)\big)} - 2(-1 - 3) = 8$$

$$a\,x = b$$

$$x = \frac{b}{a} \qquad \text{if } a \neq 0$$

1 solution

if $a = 0$:

$0 \cdot x = b$ if $b \neq 0$ no solutions

if $b = 0$ infinite solutions.

$\underline{\underline{A}}$ $n \times n$ matrix

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

if $\underline{\underline{A}}^{-1}$ exists i.e. if $\det \underline{\underline{A}} \neq 0$

$$\underline{\underline{A}}^{-1}\,\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}^{-1}\,\underline{b}$$

$$\underline{\underline{I}}\,\underline{x} = \underline{\underline{A}}^{-1}\,\underline{b}$$

$$\underline{x} = \underline{\underline{A}}^{-1}\,\underline{b}$$ 1 solution

According to what we have just written, The only hope to have solutions if $\det \underline{\underline{A}} = \underline{0}$ is to have $\underline{b} = \underline{0}$. In this case, the system becomes $\underline{\underline{A}}\,\underline{x} = \underline{0}$ (homogeneous system)

So if $\det \underline{\underline{A}} = 0$, the homogeneous system has infinite (non trivial) solutions.