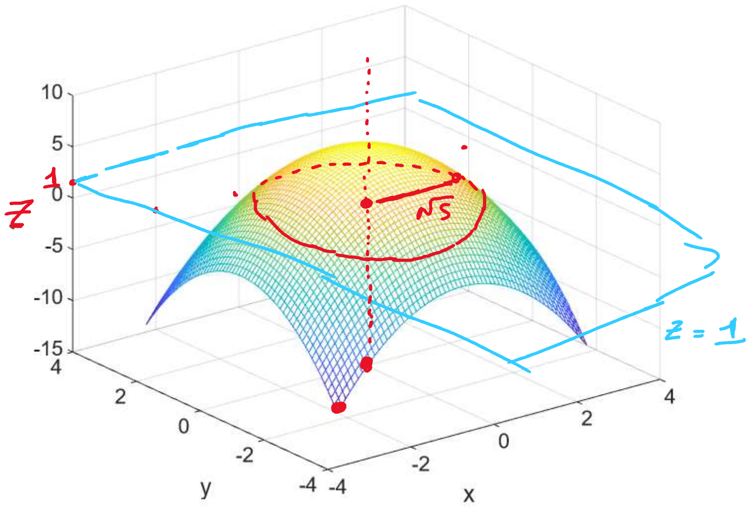
$$z = 6 - x^2 - y^2$$



$$\begin{cases} z = f(x, y) \\ z = k \end{cases}$$

$$k = 6 - x^2 - y^2$$

$$x^2 + y^2 = \underbrace{6 - k}_{r^2}$$

$$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

$$6 - k \geq 0$$

$$k \leq 6$$

$$\infty$$

QUADRATIC FORMS

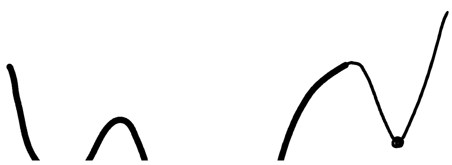
1

$$x_3 = x_1^2 + x_2^2$$

2 $$x_3 = -x_1^2 - x_2^2$$

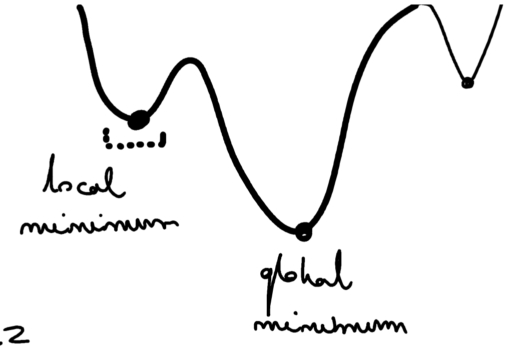
3 $$x_3 = x_1^2 - x_2^2$$

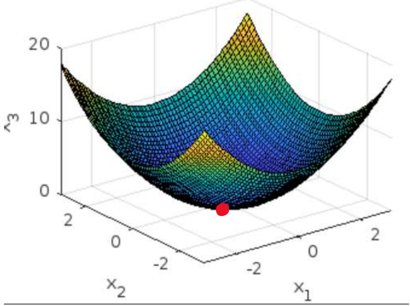
4 $$x_3 = (x_1 + x_2)^2$$



5 $$x_3 = (\quad)^2$$

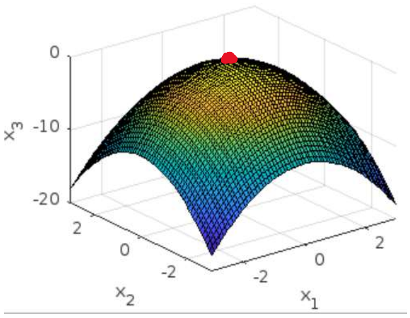
⑤ $x_3 = -(x_1 + x_2)^2$





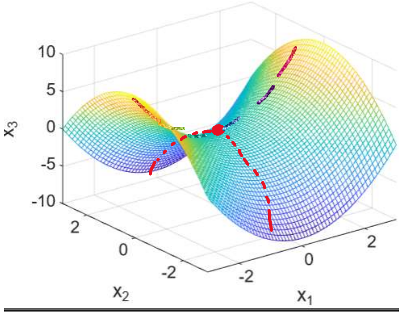
$$x_3 = \underbrace{x_1^2}_{\text{monomial of degree 2}} + x_2^2$$

The origin is a strict ~~local~~ global minimum



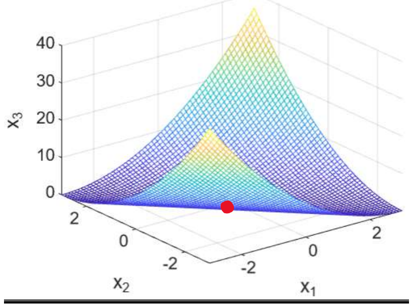
$$x_3 = -x_1^2 - x_2^2$$

The origin is a strict global maximum



$$x_3 = x_1^2 - x_2^2$$

The origin is a saddle point

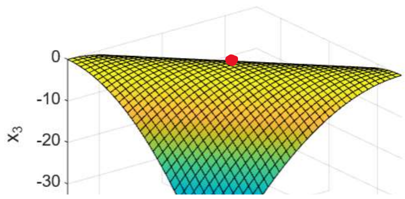


$$x_3 = (x_1 + x_2)^2$$

or

$$x_3 = x_1^2 + \underbrace{2x_1x_2} + x_2^2$$

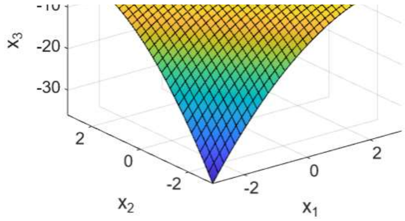
The origin is a non-strict global minimum



$$x_3 = -(x_1 + x_2)^2$$

or

$$x_3 = -x_1^2 - 2x_1x_2 - x_2^2$$



$$x_3 = -x_1^2 - 2x_1x_2 - x_2^2$$

The origin is a non-strict global maximum.

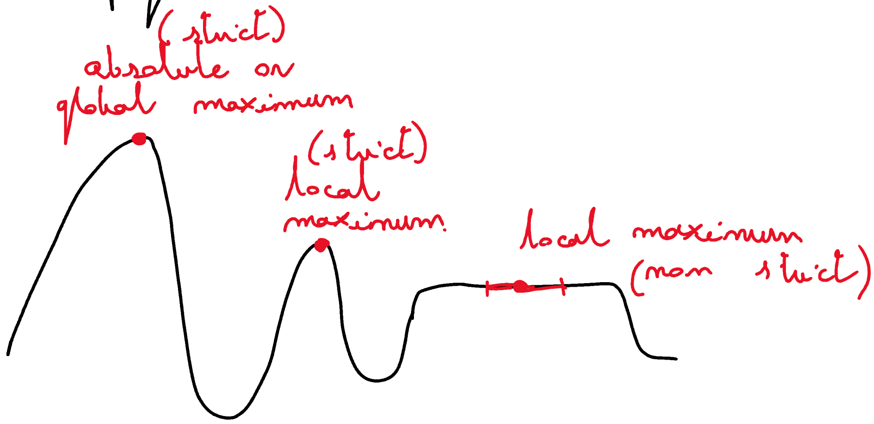
Features in common:

1) there is no constant term, i.e.

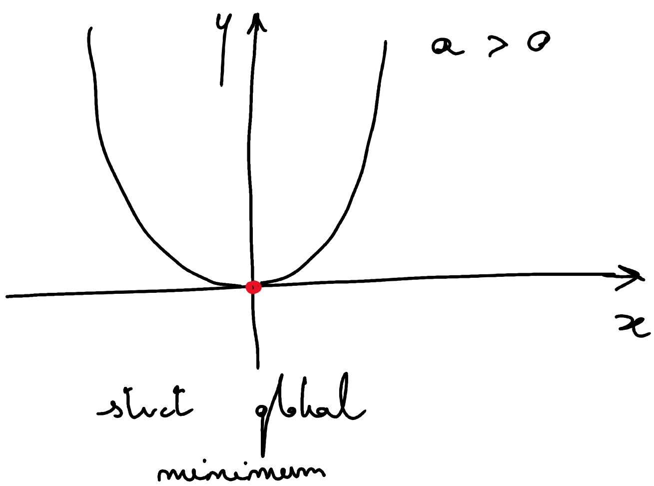
$$x_3 = \dots + 1$$

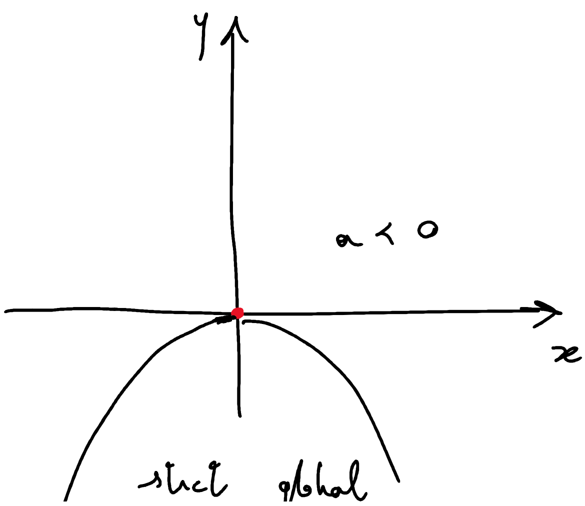
2) there are only monomials of degree two.

As a consequence, the origin belongs to all these figures.



$y = a x^2$ the simplest quadratic form, with $a \neq 0$.





strict global minimum

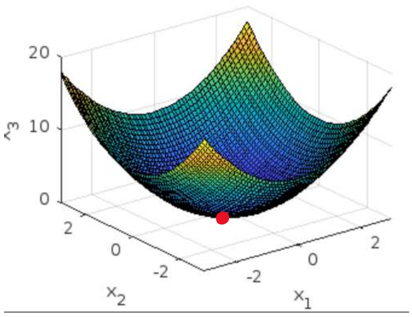
strict global maximum

General definition of quadratic form:

$$Q(x_1, x_2, \dots, x_m) = a_{11} x_1^2 + a_{12} x_1 x_2 + \dots + a_{1m} x_1 x_m + a_{21} x_2 x_1 + \dots + a_{mm} x_m^2 =$$

or

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} x_i x_j \quad \overset{\text{or}}{=} \quad \sum_{i,j} a_{ij} x_i x_j$$

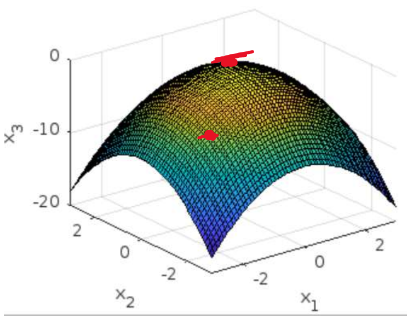


$Q(x_1, x_2)$

The quadratic form $Q(x_1, x_2)$ is said to be positive definite:

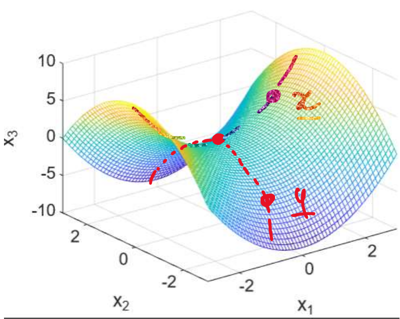
$\forall (x_1, x_2) \neq 0, \quad Q(x_1, x_2) > 0$

$\forall$ universal quantifier (read "for all")



The quadratic form $Q(x_1, x_2)$ is said to be negative definite:

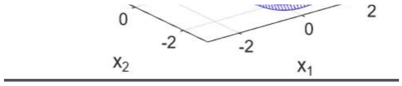
$\forall (x_1, x_2) \neq 0, \quad Q(x_1, x_2) < 0$



The quadratic form $Q(x_1, x_2)$ is said to be indefinite:

$\exists \ \underline{x}, \underline{y}$:

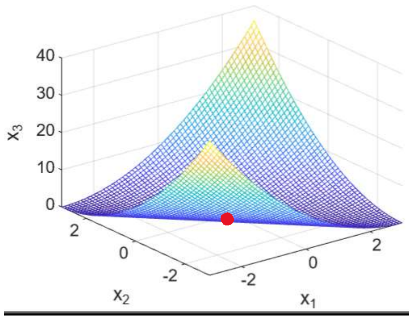
$Q(\underline{x}) > 0$ and $Q(\underline{y}) < 0$



$\exists \; \underline{x}, \underline{y}$:

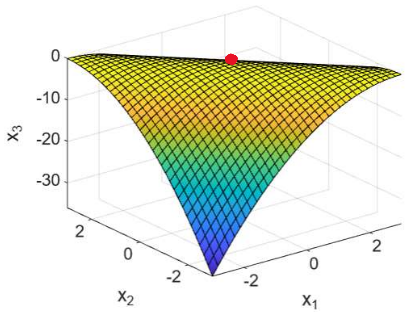
$$Q(\underline{x}) > 0 \quad \text{and} \quad Q(\underline{y}) < 0$$

$\exists$: read it as "There exists"



The quadratic form $Q(x_1, x_2)$ is said to be positive semi definite :

$$\forall (x_1, x_2) \neq 0 \qquad Q(x_1, x_2) \geq 0$$



The quadratic form $Q(x_1, x_2)$ is said to be ~~semi~~ negative semi definite :

$$\forall (x_1, x_2) \neq 0 \;, \qquad Q(x_1, x_2) \leq 0$$

$$Q(x_1, x_2) = 2x_1^2 + 4x_1x_2 + 0\,x_2x_1 + 3x_2^2 =$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, in general, we will express quadratic forms in matrix notation:

$$Q(\underline{x}) = \underline{x}^T \underline{\underline{A}}\, \underline{x}$$

$$Q(x_1, x_2) = 2x_1^2 + 4x_1x_2 + 3x_2^2 =$$

$$= 2x_1^2 + 2x_1x_2 + 2x_1x_2 + 3x_2^2 =$$

$$= 2x_1^2 + 2x_1x_2 + 2x_2x_1 + 3x_2^2 =$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

There is an infinite number of ways we can express a quadratic form in matrix notation. However, the most convenient way is to work with a matrix $\underline{\underline{A}}$ that is symmetric, that is: $\underline{\underline{A}}^T = \underline{\underline{A}}$.

Let $\underline{\underline{A}}$ be a symmetric matrix. Then $\underline{\underline{A}}$ is:

- positive definite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} > 0 \quad \forall\ \underline{x} \neq \underline{0}$
- negative definite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} < 0 \quad \forall\ \underline{x} \neq \underline{0}$
- indefinite: if $\exists\ \underline{x}, \underline{y} \neq \underline{0} \ \big|\ \underline{x}^T \underline{\underline{A}}\, \underline{x} > 0$ and $\underline{y}^T \underline{\underline{A}}\, \underline{y} < 0$

  (such that, or :)

- positive semidefinite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} \geq 0 \quad \forall\ \underline{x} \neq \underline{0}$

- negative semidefinite: if $\underline{x}^T \underline{\underline{A}}\, \underline{x} \leq 0 \quad \forall\ \underline{x} \neq \underline{0}$

Ex

$$Q(x_1, x_2, x_3) = \overbrace{2x_1^2 + 3x_2^2 - x_3^2}^{\text{don't touch them}} + \overbrace{4x_1x_3 - 6x_2x_3}^{\text{divide them by 2}} =$$

$$Q(x_1, x_2, x_3) = \overbrace{2x_1^2 + 3x_2^2 - x_3^2} + \overbrace{4x_1x_3 - 6x_2x_3} =$$

$$= 2x_1^2 + 3x_2^2 - x_3^2 + 2x_1x_3 + 2x_1x_3$$
$$- 3x_2x_3 - 3x_2x_3 =$$

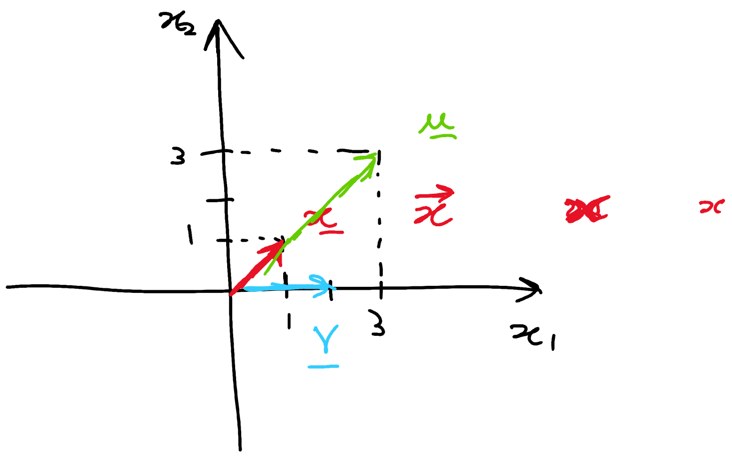
swap the indexes

$$= 2x_1x_1 + 3x_2x_2 - x_3x_3 + 2x_1x_3 + 2x_3x_1$$
$$- 3x_2x_3 - 3x_3x_2 =$$

$$= \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}}_{\underline{x}^T} \underbrace{\begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & -3 \\ 2 & -3 & -1 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}}$$

$$\infty$$

$$\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

eigenvalue

$$\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 1 & 2 \\ & \end{bmatrix} \begin{bmatrix} 1 \\ \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 1 \\ \end{bmatrix} \begin{bmatrix} 3 \\ \end{bmatrix} = \begin{bmatrix} 1 \\ \end{bmatrix}$$

$$\underline{u} = \underline{\underline{A}}\,\underline{x} = \begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}1\cdot 1+2\cdot 1\\ 0\cdot 1+3\cdot 1\end{bmatrix} = \begin{bmatrix}3\\ 3\end{bmatrix} = 3\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\underline{\underline{B}} = \begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}$$

$$\underline{v} = \underline{\underline{B}}\,\underline{x} = \begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}0\cdot 1+2\cdot 1\\ 1\cdot 1-1\cdot 1\end{bmatrix} = \begin{bmatrix}2\\ 0\end{bmatrix}$$

If we multiply a vector by a scalar, the result is a parallel vector.
But in general, if we multiply a matrix by a vector, the result is not a parallel vector.