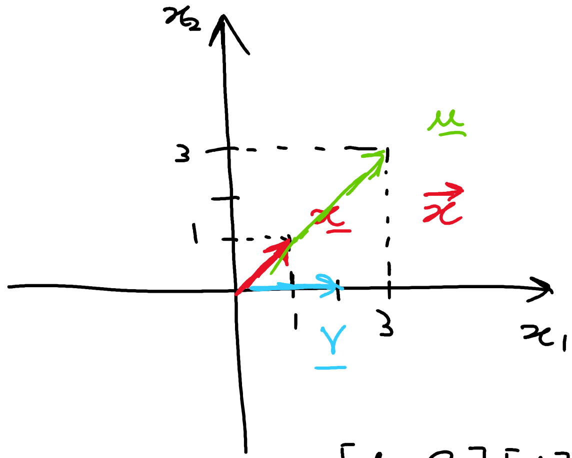


eigenvalue

$$\underline{u} = \underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 1 \\ 0\cdot 1 + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{v} = \underline{\underline{B}}\,\underline{x} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 + 2\cdot 1 \\ 1\cdot 1 - 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Let $\underline{\underline{A}}$ be a square matrix, then if there exists a vector $\underline{x}$ and a number λ such that $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$ then λ is called eigenvalue and $\underline{x}$ is called eigenvector.

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$$

$$\underline{\underline{A}}\,\underline{x} - \lambda \underline{x} = \underline{0}$$ zero vector

Identity matrix: $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$

$$a \cdot 1 = 1 \cdot a$$

$$\underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{I}}\,\underline{\underline{A}}$$

$$\underline{\underline{A}}\,\underline{x} - \lambda\,\underline{x} = \underline{0}$$

$$\underline{\underline{A}}\,\underline{x} - \lambda\,\underline{\underline{I}}\,\underline{x} = \underline{0}$$

$$\left(\underline{\underline{A}} - \lambda\,\underline{\underline{I}}\right)\underline{x} = \underline{0}$$

homogeneous linear system of equations

This system has always the trivial **0** solution. However, we are interested in the case of non trivial solutions. This holds when

$$\det\left(\underline{\underline{A}} - \lambda\,\underline{\underline{I}}\right) = 0$$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\det\left(\underline{\underline{A}} - \lambda\,\underline{\underline{I}}\right) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(4-\lambda) - 4 = 0$$

$$(\lambda-1)(\lambda-4) - 4 = 0$$

$$\lambda^2 - 5\lambda + \cancel{4} - \cancel{4} = 0$$

$$\lambda(\lambda-5) = 0$$

$\lambda = 0 \quad \vee \quad \lambda = 5$ $\qquad \cup$

vel $\quad$ or $\qquad \wedge$

aut $\qquad$ et

and

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}$$

$$(\underline{\underline{A}} - 5\underline{\underline{I}})\underline{x} = \underline{0}$$

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4x_1 + 2x_2 \\ 2x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -4x_1 + 2x_2 = 0 \\ 2x_1 - x_2 = 0 \end{cases}$$

$$\begin{cases} 2x_1 - x_2 = 0 \\ \cancel{2x_1 - x_2 = 0} \end{cases}$$

$$x_2 = 2x_1$$

that is

id est

$$x_2 = 2x_1$$

that is

id est

This equation admits infinite solutions. This is not surprising because since the very beginning we have imposed the condition to have non-trivial solutions, i.e., det(A - lambda I) = 0. This means that there are infinite vectors such that if they are multiplied by the matrix A, the result will be a vector that will be five times the original one.

For example, $x_1 = 1$, then $x_2 = 2$

$$\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ this is one eigenvector

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

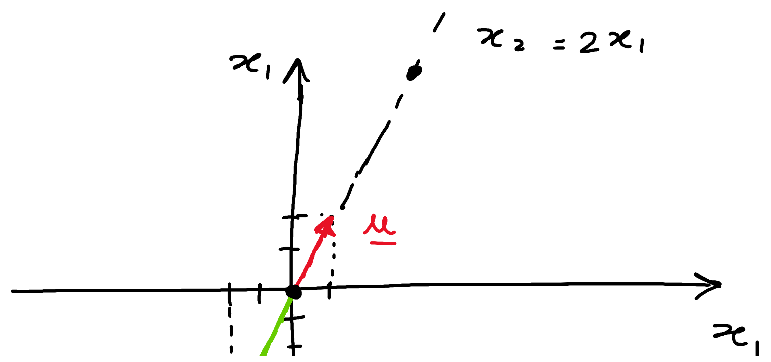
$$\underline{\underline{A}}\,\underline{u} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 \\ 2 \cdot 1 + 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5\underline{u}$$

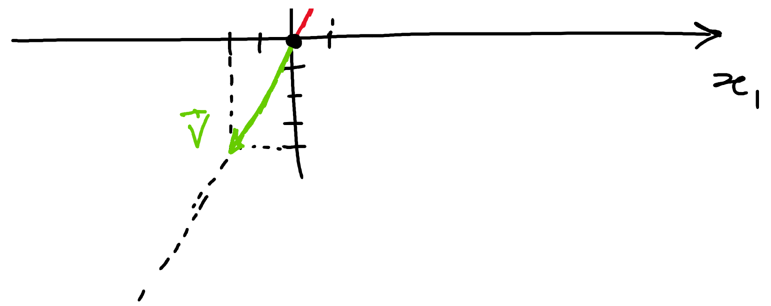
If $x_1 = -2$, then $x_2 = -4$

$$\underline{v} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{v} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1(-2) + 2(-4) \\ 2(-2) + 4(-4) \end{bmatrix} = \begin{bmatrix} -10 \\ -20 \end{bmatrix} = 5 \begin{bmatrix} -2 \\ -4 \end{bmatrix} =$$

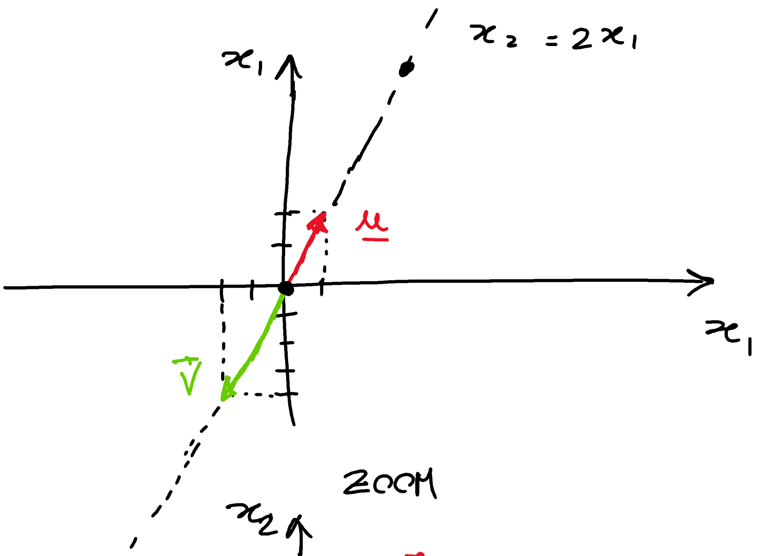
$$= 5\underline{v}$$





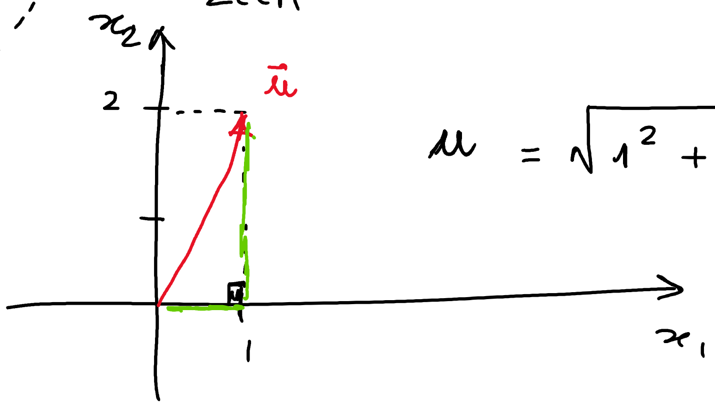
A Versor: is a vector whose modulus is equal to 1.

If you divide a non-zero vector by its modulus, you get an associated versor (with the same direction and verse).



$$\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

ZOOM



$$u = \sqrt{1^2 + 2^2} = \sqrt{5}$$

In general, if you have a vector

$\underline{x} = (x_1, x_2, \dots, x_n)$, its modulus is:

$\underline{x} = (x_1, x_2, \dots, x_n)$, its modulus is:

$$x = \sqrt{x_1^2 + x_2^2 + \dots + x_m^2}$$

This quantity is also defined as the norm of vector $\underline{x}$ and can also be denoted as

$$\|\underline{x}\|, \quad |\underline{x}|, \quad \|\vec{x}\| \quad \text{or} \quad |\vec{x}|$$

So, given a vector $\underline{x}$, its associated versor is $\dfrac{\underline{x}}{\|\underline{x}\|}$

$$\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\wedge$ circumflex accent

$$\hat{\underline{u}} = \frac{\underline{u}}{\|\underline{u}\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

normalized eigenvector

```
% Compute eigenvalues and eigenvectors
[V, L] = eig(A)
```

```
V =

  -0.8944    0.4472
   0.4472    0.8944

L =

     0         0
     0         5
```

is an eigenvector associated to the eigenvalue 5

L is a matrix having the eigenvalues on the main diagonal

Diagonalization:

$$\underline{\underline{A}} = \underline{\underline{V}}^{-1} \, \underline{\underline{L}} \, \underline{\underline{V}}$$

3) It can be proved that **if A is symmetric** then it admits only real eigenvalues.

That is also why it is better to express a quadratic form with a symmetric matrix **A.**

TH. on CLASSIFICATION OF QUADRATIC FORMS. Consider a quadratic form Q and let A be the matrix associated to Q. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A. Then Q is

Positive definite iff all the eigenvalues of A are positive,

Negative definite if all the eigenvalues of A are negative,

Positive semidefinite if all the eigenvalues of A are not negative and at least one is zero

Negative semidefinite if all the eigenvalues of A are not positive and at least one is zero

Indefinite if A admits both positive and negative eigenvalues

$$Q_1 = 2x_1^2 - 5x_2^2 + 3x_3^2, \quad =$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & -5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$$

The eigenvalues are $2, -5, 3$.

Indefinite.

Indefinite.

Ex (*)

$$Q4 = 4x_1x_2 - x_1x_3 - 6x_2x_3 =$$

$$= 2x_1x_2 + 2x_2x_1 - \frac{1}{2}x_1x_3 - \frac{1}{2}x_3x_1 - 3x_2x_3 - 3x_3x_2 =$$

$$[x_1\ x_2\ x_3]\begin{bmatrix} 0 & 2 & -\frac{1}{2} \\ 2 & 0 & -3 \\ -\frac{1}{2} & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

```
ans =

   -3.3881
   -0.4602
    3.8483
```

indefinite.

Definition: let **A** be an n x n matrix. A k x k submatrix of **A** formed by deleting n - k columns, say columns i_1, i_2, ... i_{n - k} and the same n - k rows from **A** is called a k-th order principal submatrix of **A**. The determinant of a k x k principal submatrix is called a k - th order principal minor of **A**.

i_1 I mean i_1

i_{n-k} i_{n-k}

$$\underline{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$$

$n = 4$

For example, I choose $k = 2$

I choose column 1 and 4

I choose column 1 and 4

$i_1 = 1$

$i_2 = 4$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}^{T}$$

The result is the submatrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

This is a 2^{nd} order principal submatrix of $\underline{\underline{A}}$

$\det \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 1 \cdot 2 - 2 \cdot 1 = 0$ this is a 2^{nd} order principal minor of $\underline{\underline{A}}$.