Definition: let **A** be an n x n matrix. A k x k submatrix of **A** formed by deleting n - k columns, say columns i_1, i_2, ... i_{n - k} and the same n - k rows from **A** is called a k-th order principal submatrix of **A**. The determinant of a k x k principal submatrix is called a k - th order principal minor of **A**.

EX :

3 - rd order square matrix

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

1 - st order principal submatrices , $K = 1$

I have to delete $n - K = 3 - 1 = 2$ columns and the same rows.

Delete columns 1 and 2

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ \cancel{a_{31}} & \cancel{a_{32}} & a_{33} \end{bmatrix} \qquad [a_{33}]$$

Delete columns 1 and 3

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & \cancel{a_{23}} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} \end{bmatrix} \qquad [a_{22}]$$

Delete columns 2 and 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The result: $[a_{11}]$

2 nd order $k = 2$

I have to delete $m - k = 3 - 2 = 1$ column and the corresponding row.

I delete column 1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

delete column 2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

delete column 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

How about if $k = 3$

3-rd order principal submatrix

I have to delete $n - k = 3 - 3 = 0$ columns

i.e. it is the matrix $\underline{\underline{A}}$ itself and its determinant is the only 3-rd order principal minor.

Let **A** be an n x n matrix. The k-th order principal submatrix of A obtained by deleting the last n - k rows and the last n - k columns from **A** is called the k-th order leading principal submatrix of **A**. Its determinant is called the k-th order leading principal minor of **A**.

1-st order leading principal submatrix and minor

Delete the last n - k rows

i.e. $3 - 1 = 2$ rows

$$\begin{bmatrix} a_{11} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} \end{bmatrix}$$

$$\underline{\underline{A}}_1 = [a_{11}]$$

$$|\underline{\underline{A}}_1| = a_{11}$$ 1-st order leading principal minor

2-nd order leading principal submatrix and minor

$n - k = 3 - 2 = 1$

$$\begin{bmatrix} a_{11} & a_{12} & \cancel{a_{13}} \\ a_{21} & a_{22} & \cancel{a_{23}} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} \end{bmatrix}$$

$$\underline{\underline{A}}_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|\underline{\underline{A}}_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

3-rd order leading principal submatrix

$$\underline{\underline{A}}_3 = \underline{\underline{A}}$$

$$|\underline{\underline{A}}_3| = \det \underline{\underline{A}}$$

Practical rule:

$\underline{\underline{A}}_1$ $\quad$ $\underline{\underline{A}}_2$ $\quad$ $\underline{\underline{A}}_3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{bmatrix}$$

This technique is also known as north-west minors rule

"Complete the square"

$$x^2 + y^2 - 2x - 4y - 4 = 0$$

$$x^2 - 2x + y^2 - 4y - 4 = 0$$

$$x^2 - 2 \cdot 1 \cdot x + y^2 - 2 \cdot y \cdot 2 - 4 = 0$$

$$\underbrace{x^2 - 2x + 1^2} + \underbrace{y^2 - 4y + 2^2} - 4 = 1^2 + 2^2$$

$$(x-1)^2 + (y-2)^2 - 4 = 5$$

$$(x-1)^2 + (y-2)^2 = 9$$

Circle with center $(1,2)$ and radius 3

$$(x-\alpha)^2+(y-\beta)^2=r^2$$

$$\sim$$

$$ax^2+bx+c=0 \qquad a\neq 0$$

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0$$

$$\overbrace{x^2+2\frac{b}{2a}x}+\frac{c}{a}=0$$

$$A^2+2A\cdot B$$

$$(A+B)^2=A^2+2AB+B^2$$

$$\underbrace{x^2+2x\frac{b}{2a}}$$

$$\underbrace{x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}}+\frac{c}{a}=\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$Q(x_1, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

2-nd order
quadratic form

$$Q(x_1, x_2) = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} x_i x_j =$$

$$= a_{11} x_1^2 + \underbrace{a_{12} x_1 x_2 + a_{21} x_2 x_1}_{(a_{12} + a_{21}) x_1 x_2} + a_{22} x_2^2$$

It's not a lack of generality if I rename this quadratic form as:

$$Q(x_1, x_2) = a x_1^2 + 2b x_1 x_2 + c x_2^2$$

$$Q(x_1, x_2) = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} x_i x_j =$$

$$= a_{11} x_1^2 + a_{12} x_1 x_2 + a_{21} x_2 x_1 + a_{22} x_2^2 =$$

$$= a_{11} x_1^2 + (a_{12} + a_{21}) x_1 x_2 + a_{22} x_2^2$$

$$Q(x_1, x_2) = a x_1^2 + 2b x_1 x_2 + c x_2^2 =$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad a \neq 0$$

$$Q(x_1, x_2) = a x_1^2 + 2 b x_1 x_2 + c x^2 =$$

$$= a\left(x_1^2 + 2 x_1 \frac{b}{a} x_2\right) + c x_2^2 =$$

$$= a\left(x_1^2 + 2 x_1 \frac{b}{a} x_2 + \frac{b^2}{a^2} x_2^2 - \frac{b^2}{a^2} x_2^2\right) + c x_2^2 =$$

$$= a\left[\left(x_1 + \frac{b}{a} x_2\right)^2 - \frac{b^2}{a^2} x_2^2\right] + c x_2^2 =$$

$$= a\left(x_1 + \frac{b}{a} x_2\right)^2 + \left(c - \frac{b^2}{a}\right) x_2^2 =$$

$$= a\left(x_1 + \frac{b}{a} x_2\right)^2 + \frac{ac - b^2}{a} x_2^2$$

If $x_1 \neq 0$ and $x_2 \neq 0$ this term is always positive

if $x_2 \neq 0$ x_2^2 is always positive

Then $Q(x_1, x_2) > 0$ for $(x_1, x_2) \neq (0,0)$
if $a > 0$ and $ac - b^2 > 0$

To summarize, if a > 0 and ac - b^2 > o then, independently of x1 and x2, we will have that Q(x1, x2) will be always positive provided that x1 and x2 are not zero. But this means that Q(x1, x2) is positive definite.

$$\underline{A} = \begin{bmatrix} \boxed{a} & b \\ b & c \end{bmatrix}$$

But a is the 1-st order leading principal minor
and $ac - b^2$ is the 2-nd order leading principal minor.

$$Q(x_1, x_2) = a\underbrace{\left(x_1 + \frac{b}{a}x_2\right)^2}_{>0} + \frac{ac - b^2}{\underbrace{a}_{<0}}\underbrace{x_2^2}_{>0}$$

$x_1, x_2 \neq 0$

We want $Q(x_1, x_2) < 0$ for $(x_1, x_2) \neq (0, 0)$
For sure, it must be $a < 0$
And then it must be $ac - b^2 > 0$

Theorem: let A be an n x n symmetric matrix. Then,

a) A is positive definite if and only if all its n leading principal minors are strictly positive
b) A is negative definite if and only if its n leading principal minors alternate in sign as follows:
   |A1| < 0, |A2| > 0, |A3| < 0, etc.
   that is the k-th order leading principal minor should have the sign of (-1)^k $(-1)^k$
c) If some k-th order leading principal minor of A (or some pair of them) is nonzero but does not fit either of the above two sign patterns, then A is indefinite.

$|A_1| > 0 \qquad |A_2| > 0 \qquad |A_3| < 0$ indefinite

Ex:

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

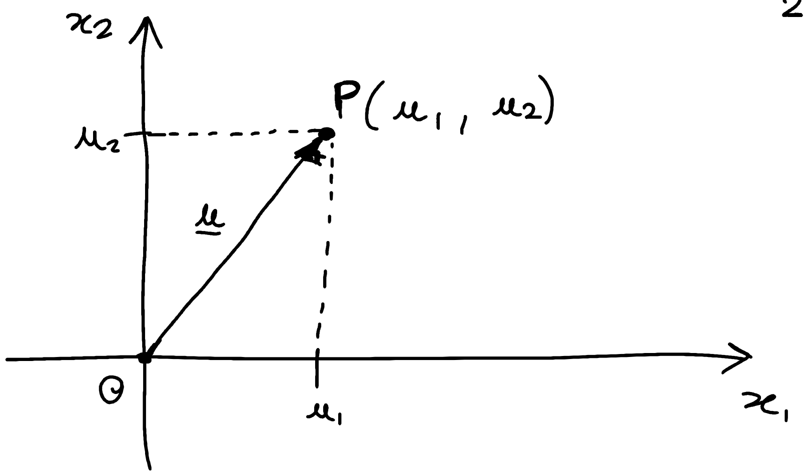
$$|A_1| = 1 > 0$$

$$|A_2| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2 < 0$$

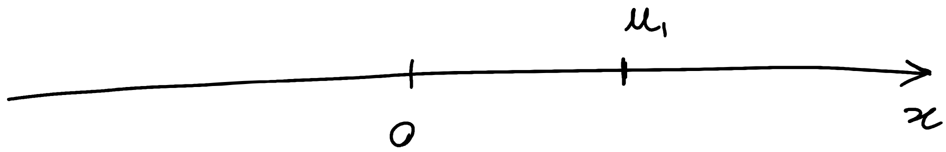
indefinite

```
>> A = [1 2; 3 4]

A =

     1     2
     3     4

>> eig(A)

ans =

   -0.3723
    5.3723
```

2 dimensions

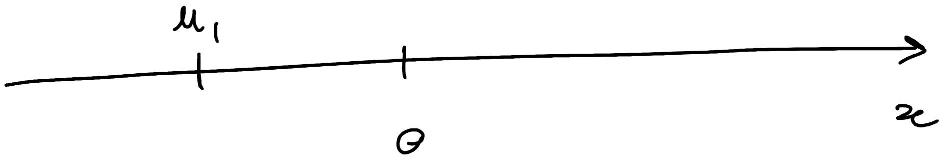


$$d(P, O) = \sqrt{u_1^2 + u_2^2} = \|\underline{u}\|$$



distance from the origin: $u_1 - O = u_1$

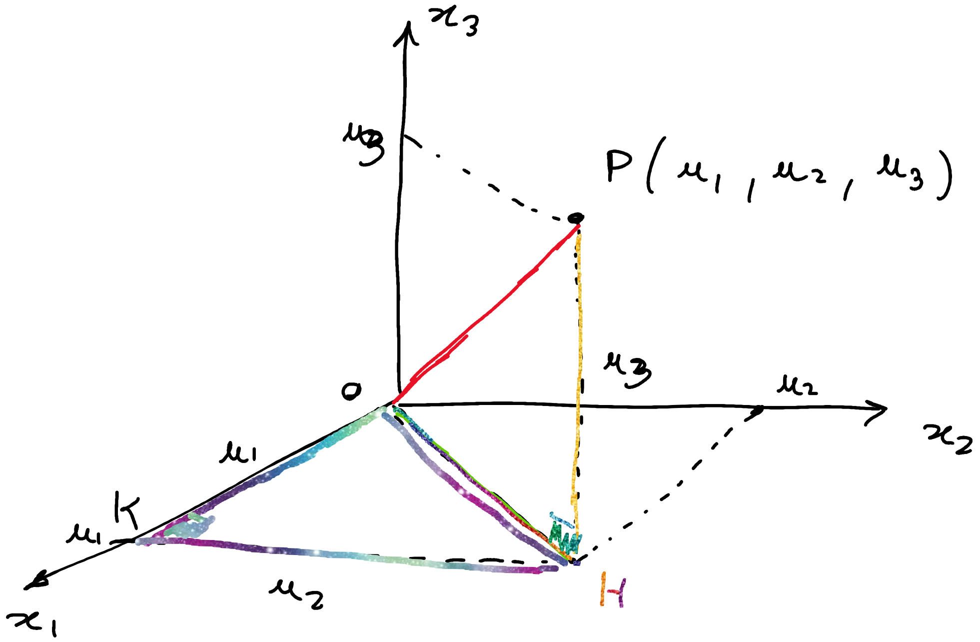
1 dimension



distance from the origin: $-u_1$

In general the distance from the origin is $|u_1|$

3 dimensions



The triangle PHO is ~~rectangle~~ right angle triangle

$$d(P,O) = \sqrt{\overline{OH}^2 + \overline{PH}^2} = \sqrt{\overline{OH}^2 + u_3^2}$$

But also OKH is ~~rectangle~~ a right angle triangle so

$$\overline{OH}^2 = u_1^2 + u_2^2$$

so

$$d(P,O) = \sqrt{u_1^2 + u_2^2 + u_3^2} = \| \underline{u} \|$$

Of course this can be generalized to n dimensions:

$$d(P,O) = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2} = \| \underline{u} \|$$

If $n = 1$

$$d(P,O) = \sqrt{u_1^2} = |u_1| = \| \underline{u} \|$$

$$d(P, Q) = \sqrt{u_1^2} = |u_1| = \|\underline{u}\|$$