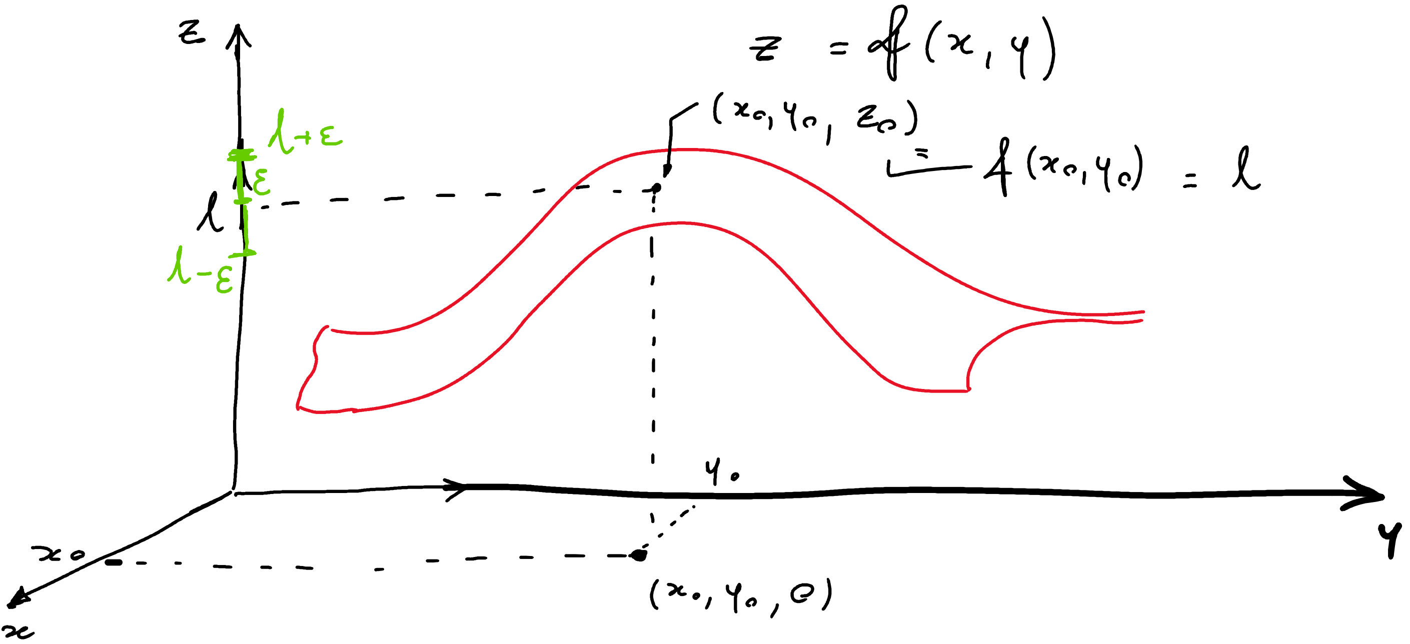
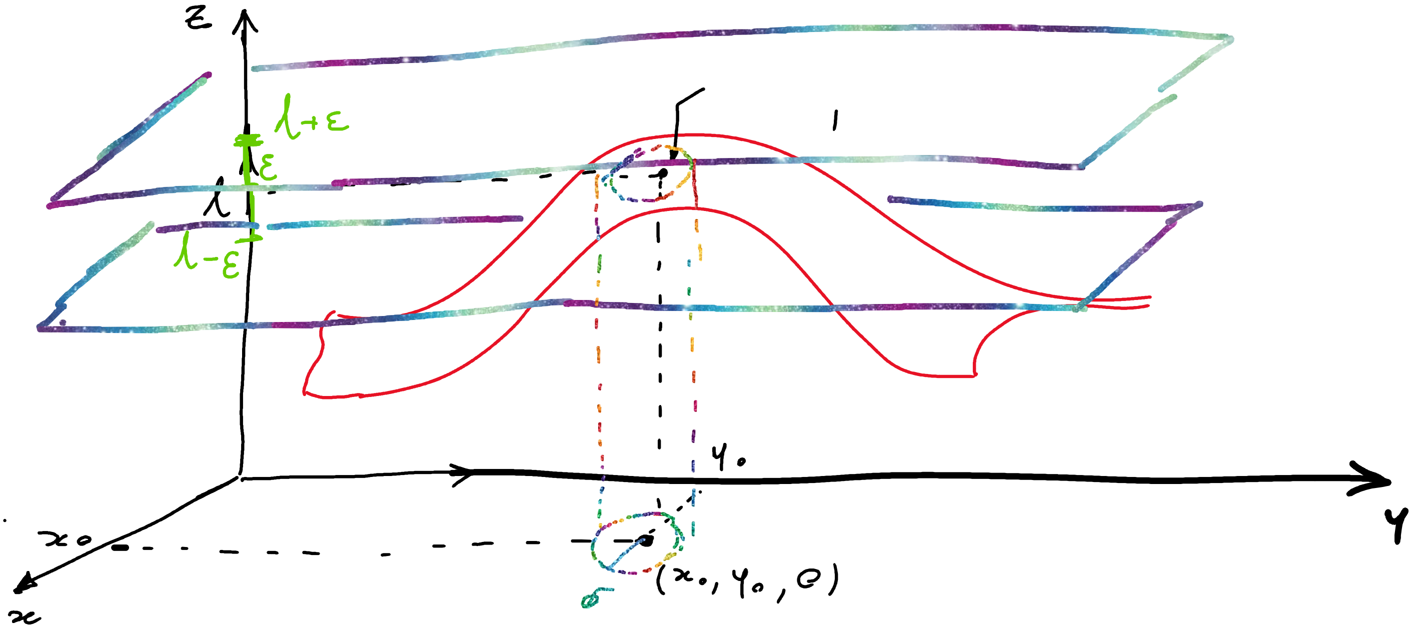
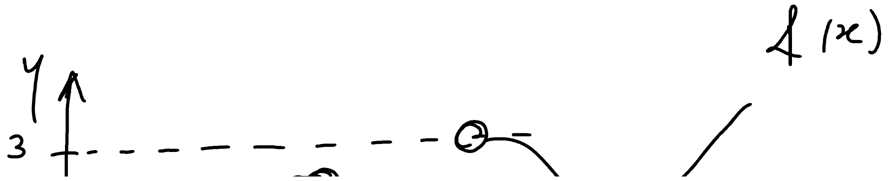
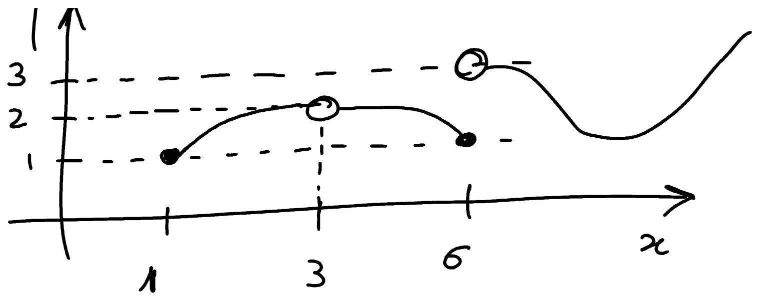


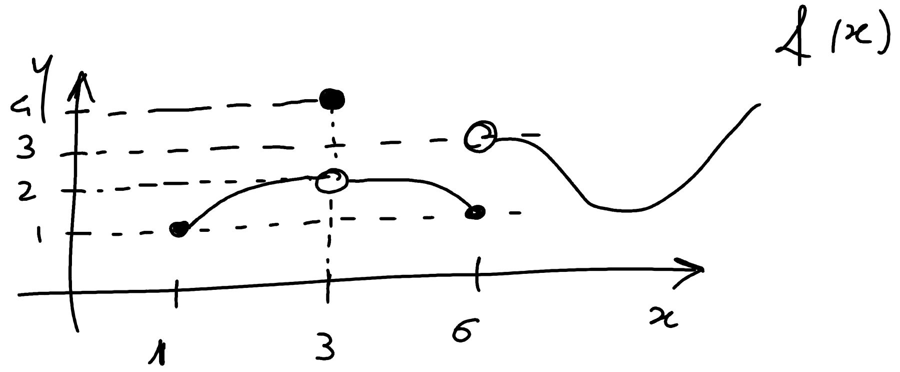
$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = l \qquad \text{or} \qquad \lim_{(x,y)\to(x_0,y_0)} f(x,y) = l$$





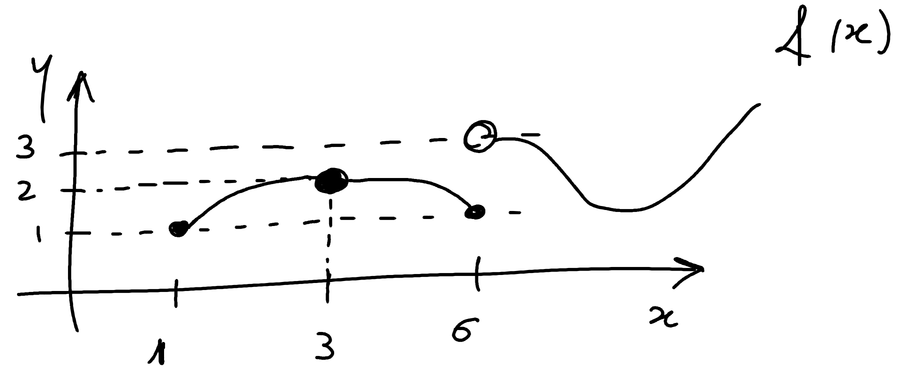


$\lim_{x \to 3} f(x) = 2$ but f is not defined in $x = 3$



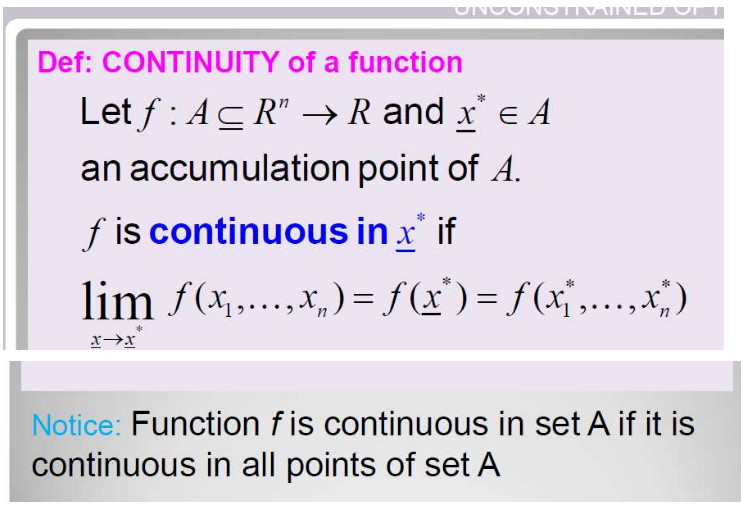
$\lim_{x \to 3} f(x) = 2$ but $f(3) = 4$

$\lim_{x \to 3} f(x) \neq f(3)$ the function is discontinuous



$\lim_{x \to 3} f(x) = 2$ but this is also equal to $f(3)$

The function is continuous at $x = 3$



Def: CONTINUITY of a function

Let $f : A \subseteq R^n \to R$ and $\underline{x}^* \in A$

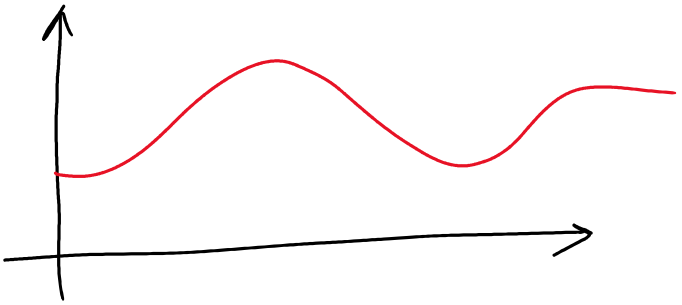
an accumulation point of A.

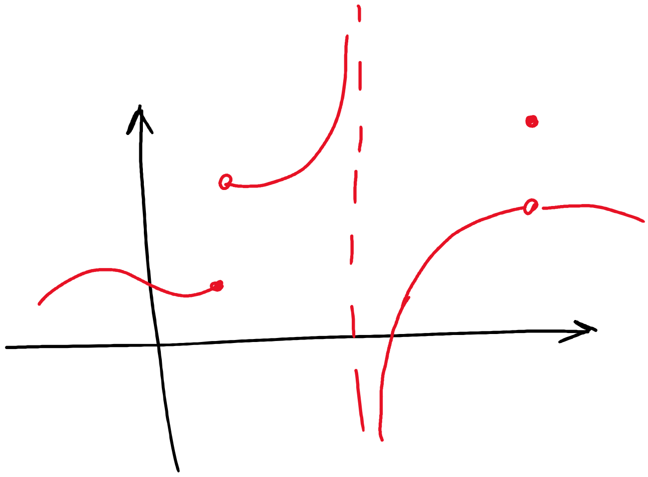
f is **continuous in $\underline{x}^*$** if

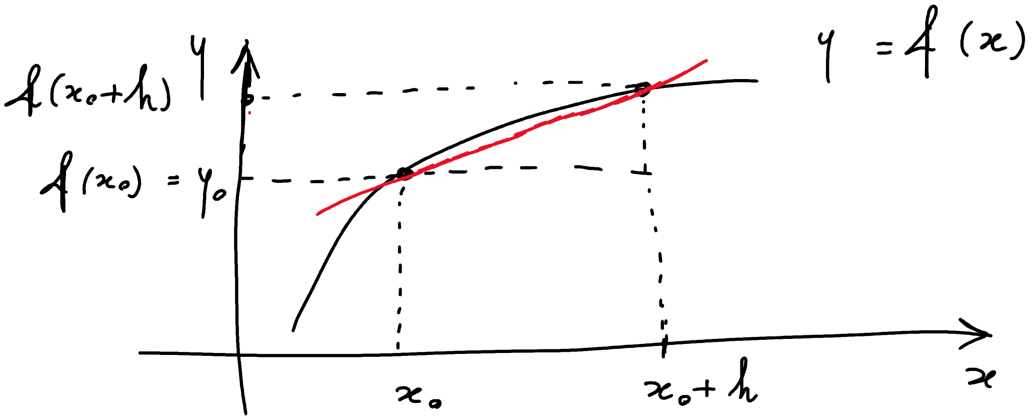
$$\lim_{\underline{x}\to\underline{x}^*} f(x_1,\dots,x_n) = f(\underline{x}^*) = f(x_1^*,\dots,x_n^*)$$

Notice: Function f is continuous in set A if it is continuous in all points of set A

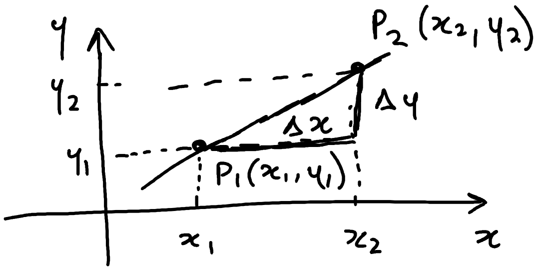
Roughly speaking, a function is continuous if I can draw it without leaving the pen from the paper



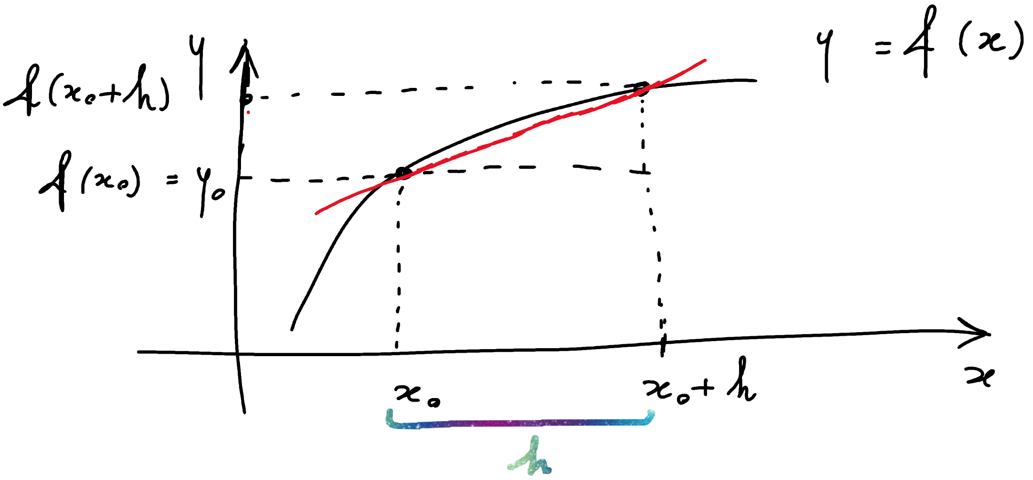




$$y = \underbrace{m}_{\text{slope}}\, x + \underbrace{q}_{\text{intercept}}$$



$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

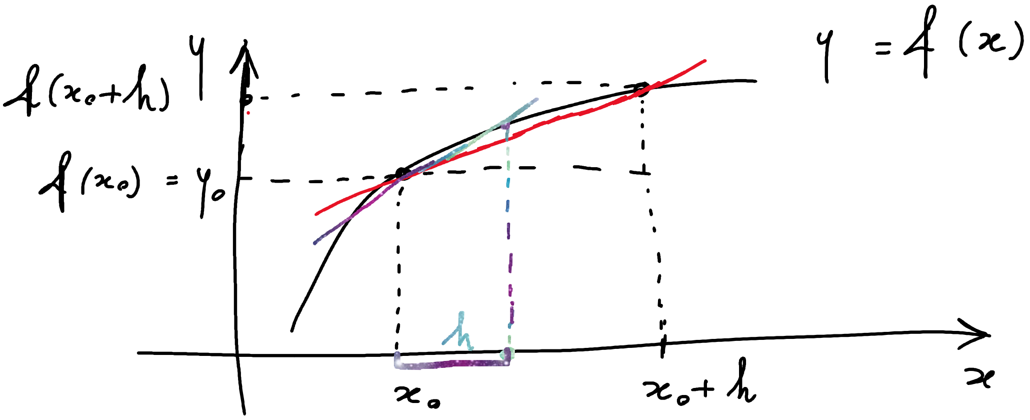


The slope of the red line is:

$$m = \frac{f(x_0+h) - f(x_0)}{x_0 + h - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$$

and it is also called incremental ratio.

Now suppose to reduce h



If I keep reducing h, the lines tends to be tangent to the function f at point (x0, f(x0)). This is the concept of derivative of a function in a point: it is the slope of the tangent line at (x0, f(x0)). That is:

$$m = f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$'$ $\quad h \to 0 \quad h$

provided that this limit exists.

If I want to know the slope of the tangent line in a generic point x, I have a function that is the first derivative of f

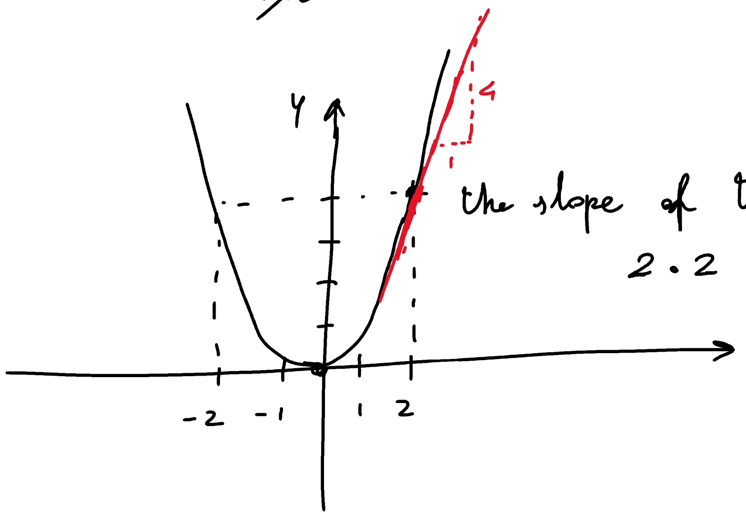
$$f'(x)$$

$$f(x) = x^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

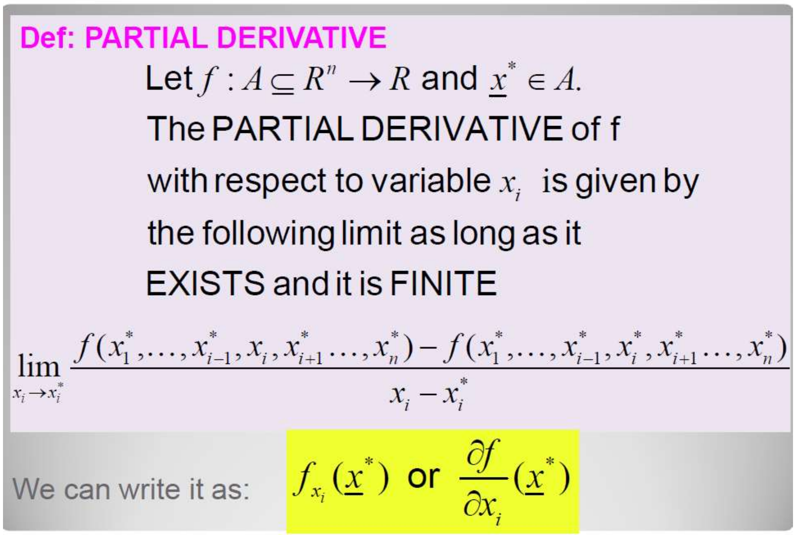
$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} (2x+h) = 2x$$



the slope of the tangent line is

$2 \cdot 2 = 4$



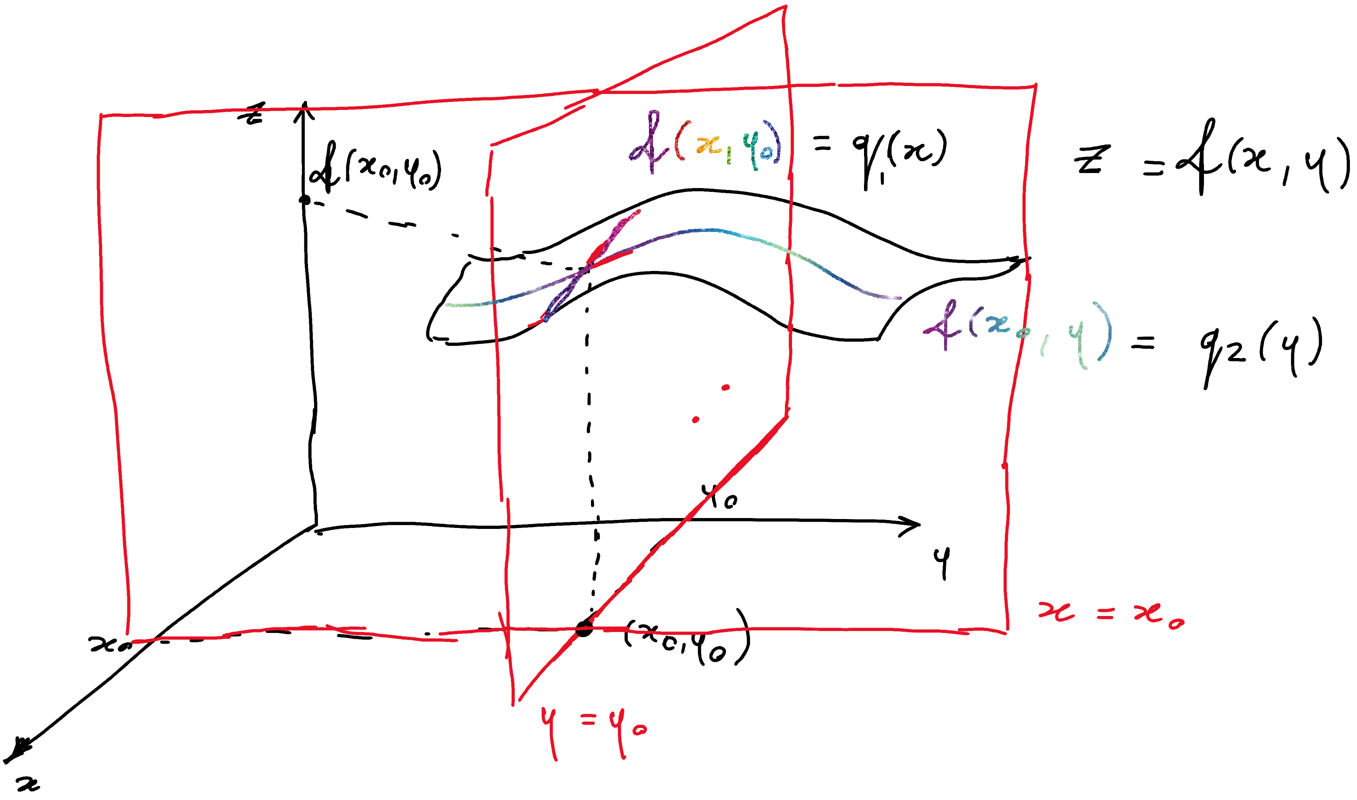
Def: PARTIAL DERIVATIVE

Let $f : A \subseteq R^n \to R$ and $\underline{x}^* \in A$.

The PARTIAL DERIVATIVE of f with respect to variable x_i is given by the following limit as long as it EXISTS and it is FINITE

$$\lim_{x_i \to x_i^*} \frac{f(x_1^*,\dots,x_{i-1}^*,x_i,x_{i+1}^*\dots,x_n^*) - f(x_1^*,\dots,x_{i-1}^*,x_i^*,x_{i+1}^*\dots,x_n^*)}{x_i - x_i^*}$$

We can write it as: $f_{x_i}(\underline{x}^*)$ or $\frac{\partial f}{\partial x_i}(\underline{x}^*)$



$$f'_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f'_y(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

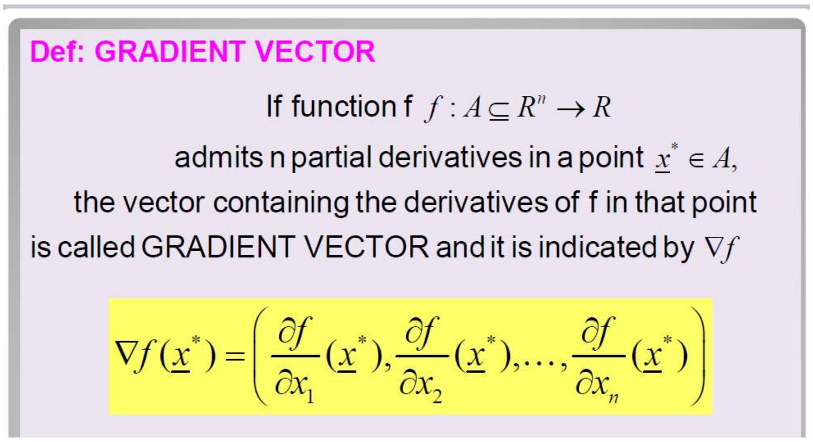
IGNORE THIS

$\underline{u}$

$$\lim_{t \to 0} \frac{f(x_0 + t u_x, y_0 + t u_y) - f(x_0, y_0)}{t}$$

$$f'_x(x_0, y_0) \quad \text{or} \quad \frac{\partial f}{\partial x}(x_0, y_0)$$

$$f'_y(x_0, y_0) \quad \text{or} \quad \frac{\partial f}{\partial y}(x_0, y_0)$$



Def: GRADIENT VECTOR

If function f $f : A \subseteq R^n \to R$

admits n partial derivatives in a point $\underline{x}^* \in A$,

the vector containing the derivatives of f in that point

is called GRADIENT VECTOR and it is indicated by ∇f

$$\nabla f(\underline{x}^*) = \left(\frac{\partial f}{\partial x_1}(\underline{x}^*), \frac{\partial f}{\partial x_2}(\underline{x}^*), \ldots, \frac{\partial f}{\partial x_n}(\underline{x}^*) \right)$$

$$z = x^2 + 2y^2 + 5x^1 y$$

- compute the gradient in general
- " " " at point (1, 2)

$$f'_x = 2x + 5y$$

$$f'_y = 4y + 5x$$

$$[2x + 5y]$$

$$y = x^\alpha$$

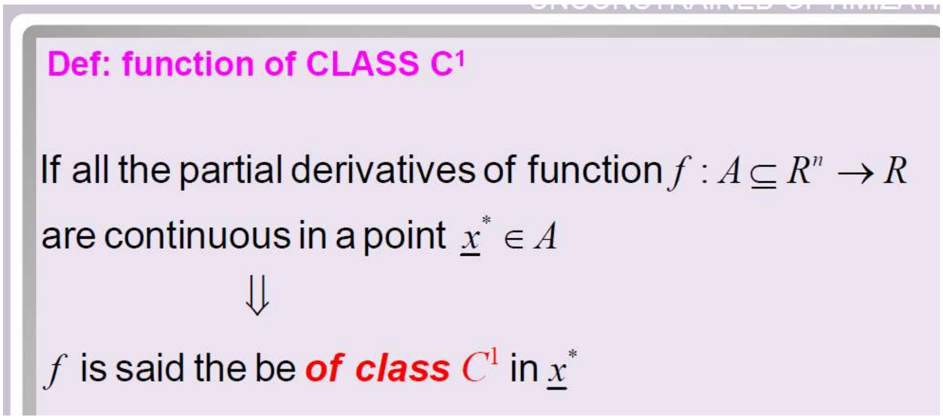
$$y' = \alpha x^{\alpha - 1}$$

$$y = k$$

$$\nabla f = \begin{bmatrix} 2x + 5y \\ 4y + 5x \end{bmatrix}$$

$y = k$
$y' = 0$

$$\nabla f(1,2) = \begin{bmatrix} 2 \cdot 1 + 5 \cdot 2 \\ 4 \cdot 2 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

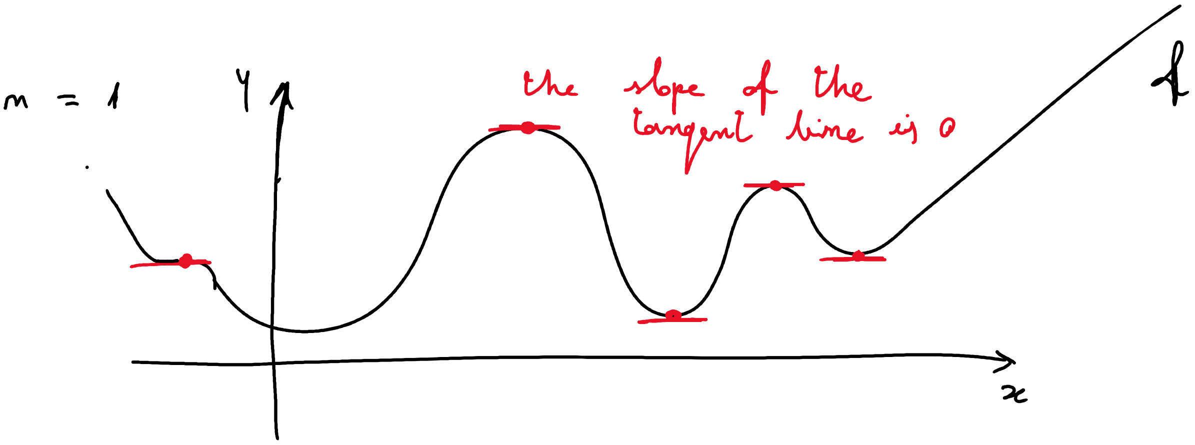


Def: function of CLASS C¹

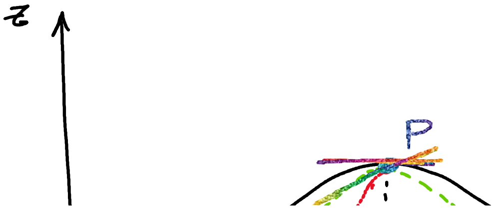
If all the partial derivatives of function $f : A \subseteq R^n \rightarrow R$ are continuous in a point $\underline{x}^* \in A$

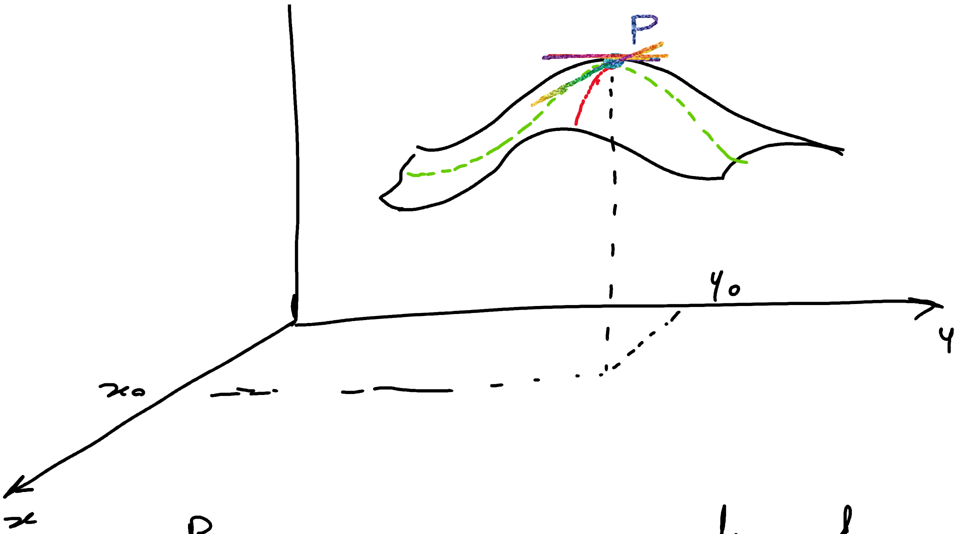
$$\Downarrow$$

f is said the be ***of class*** C^1 in $\underline{x}^*$



$f'(x) = 0$ necessary condition to determine maxima and or minima





P is a maximum also for the red and the green functions. Thus, the slope of their tangent lines at (x_0, y_0) must be 0

So $f'_x(x_0, y_0) = 0$ and $f'_y(x_0, y_0) = 0$

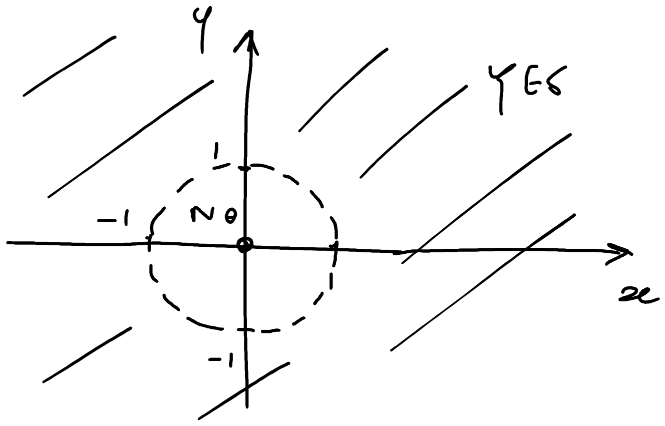
First - order necessary conditions:

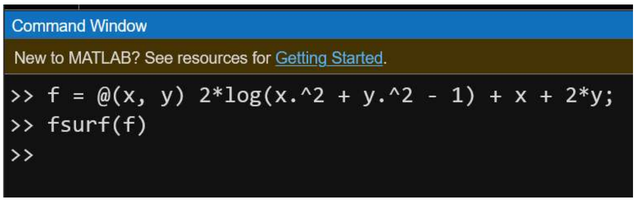
$$\boxed{\nabla f(x,y) = \underline{0}}$$

$$f(x,y) = 2\ln(x^2 + y^2 - 1) + x + 2y$$

dom f: $x^2 + y^2 - 1 > 0$

$x^2 + y^2 > 1$

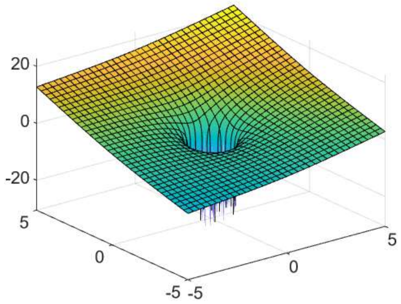




Command Window

New to MATLAB? See resources for Getting Started.

```
>> f = @(x, y) 2*log(x.^2 + y.^2 - 1) + x + 2*y;
>> fsurf(f)
>>
```



$$f(x,y) = 2\ln(\underline{x^2+y^2-1}) + x + 2y$$

$$\nabla f(x,y) = \underline{0}$$

$$f'_x = 2\,\frac{1}{x^2+y^2-1}\,2x + 1 = \frac{4x}{x^2+y^2-1} + 1$$

$$f'_y = 2\,\frac{1}{x^2+y^2-1}\,2y + 2 = \frac{4y}{x^2+y^2-1} + 2$$

$$\begin{cases} \dfrac{4x}{x^2+y^2-1} + 1 = 0 \\ \dfrac{\overset{2}{\cancel{4}}y}{x^2+y^2-1} + \overset{1}{\cancel{2}} = 0 \end{cases}$$

$$\begin{cases} \dfrac{4x}{x^2+y^2-1} = -1 \\ \dfrac{2y}{x^2+y^2-1} = -1 \end{cases}$$

$$\begin{cases} 4x = -x^2 - y^2 + 1 \\ 2y = -x^2 - y^2 + 1 \end{cases}$$

$$\begin{cases} 4x = -x^2 - y^2 + 1 \\ 2y = 4x \end{cases}$$

$$\begin{cases} 4x = -x^2 - 4x^2 + 1 \\ y = 2x \end{cases}$$

$$\begin{cases} 5x^2 + 4x - 1 = 0 \\ y = 2x \end{cases}$$

$$\begin{cases} x = -1 \\ y = -2 \end{cases} \quad \checkmark \qquad\qquad \begin{cases} x = \frac{1}{5} \\ y = \frac{2}{5} \end{cases}$$

Two candidates maximum or minimum point :

$A(-1, -2)$ and ~~$B\left(\frac{1}{5}, \frac{2}{5}\right)$~~

Please check that these points are in the domain

$$x^2 + y^2 > 1$$

$$(-1)^2 + (-2)^2 > 1$$

$$5 > 1 \quad \checkmark$$

$$\left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 > 1$$

$$\frac{1}{25} + \frac{4}{25} > 1$$

$$\frac{1}{5} > 1 \quad \text{NO}$$