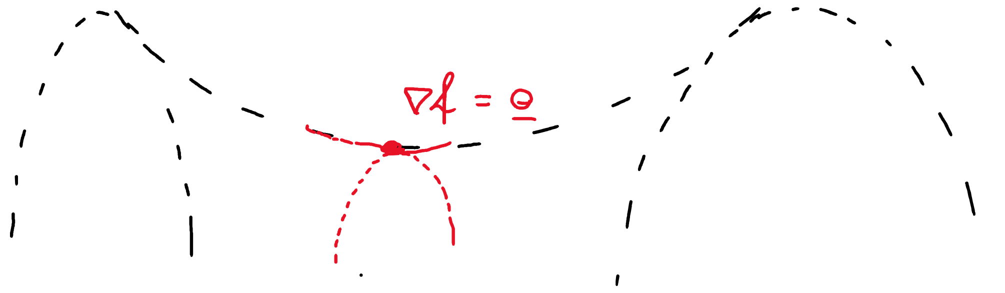


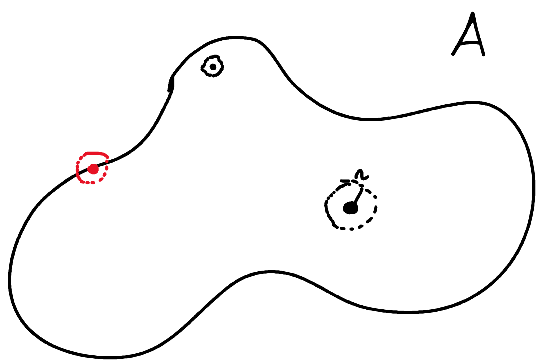
$y = f(x)$

$f'(x) = 0$ first-order (necessary) condition

$f''(x)$ second-order (sufficient) condition

Def. INTERIOR POINT

A point $\underline{x}^*$ is an interior point of A if there exists a whole r-ball about $\underline{x}^*$ in the domain A.



This set is not open because there is at least one point on the border that is not an interior point.

Open set: A is open if all its points are interior points

Def: Critical point

An interior point $\underline{x}$ is said to be a **critical point** if for all i

$$\frac{\partial f}{\partial x_i}(\underline{x}) = 0$$

A function is of class C if it is continuous
A function is of class C^1 if it is continuous with its first-order partial derivatives

Ex :

$$f(x, y) = x^2 + 3xy - y^2 x$$

$$f'_x = 2x + 3y - y^2$$

$$f'_y = 3x - 2yx$$

f is C

$f \in C$

f is C^1

$$f$$

$\frac{\partial f}{\partial x}$ same as f'_x $\qquad$ f'_y same as $\frac{\partial f}{\partial y}$

f''_{xx} $\quad f''_{xy}$ $\quad f''_{yx}$ $\quad f''_{yy}$

f''_{xy} first derive with respect to x and then with respect to y

It is the same as

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f'_x = 2x + 3y - y^2$$

$$f'_y = 3x - 2yx$$

$$f''_{xx} = 2$$

$$f''_{xy} = \underline{3 - 2y}$$

f is of class C^2

$$f''_{yy} = -2x$$

$$f''_{yx} = \underline{3 - 2y}$$

THE SECOND PARTIAL DERIVATIVES

If all the first partial derivatives are derivable again, then it is possible to calculate their partial derivatives thus obtaining:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i x_j} \quad \text{with } i \neq j$$

Mixed second derivative

$$\frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_i^2} = f_{x_i x_i}$$

Pure second derivative

Def: function of CLASS C^2

If all the second derivatives of f exist and are continuous, then f is said to be of C^2 class

We will consider only C^2 functions!

Schwarz THEOREM

If $f: A \subseteq R^n \to R$, A open set, is a C^2 function on A then

$$\Downarrow$$

$\forall \underline{x} \in A$ and $\forall\, i, j$

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\underline{x}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x})$$

Def: Hessian of f in point $\underline{x}^*$

Let f be of C^2 class and let $\underline{x}^*$ be an interior fixed point. The hessian of f in point $\underline{x}^*$ is given by:

$$Hf(\underline{x}^*) = \begin{pmatrix} f_{x_1x_1}(\underline{x}^*) & f_{x_1x_2}(\underline{x}^*) & \cdots & f_{x_1x_n}(\underline{x}^*) \\ f_{x_2x_1}(\underline{x}^*) & f_{x_2x_2}(\underline{x}^*) & \cdots & f_{x_2x_n}(\underline{x}^*) \\ \vdots & \vdots & \cdots & \vdots \\ f_{x_nx_1}(\underline{x}^*) & f_{x_nx_2}(\underline{x}^*) & \cdots & f_{x_nx_n}(\underline{x}^*) \end{pmatrix}$$

Notice that Hf is a symmetric square matrix (nxn).

$$f(x, y) = x^2 + 3xy - y^2 x$$

$$f''_{xx} = 2$$

$$f''_{xy} = \underline{3 - 2y}$$

$$f''_{yy} = -2x$$

$$f''_{yx} = \underline{3 - 2y}$$

$$\underline{\underline{H}}f(x, y) = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 3 - 2y \\ 3 - 2y & -2x \end{bmatrix}$$

TAYLOR POLYNOMIALS

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ (with A open) be a C^{k+1} function. For any $x, x_0 \in A$ there exists a point c^* between x and x_0 such that:

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots + \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k}_{} + \underbrace{\frac{1}{(k+1)!} f^{(k+1)}(c^*)(x - x_0)^{k+1}}_{}$$

$$f(x) = \boxed{\text{polynomial}} + \underbrace{\text{remainder}}_{}$$

with $\displaystyle \lim_{x \to x_0} \frac{R_k(x, x_0)}{|x - x_0|^k} = 0$

$f(x) = \sqrt{x}$ and $x_0 = 1$

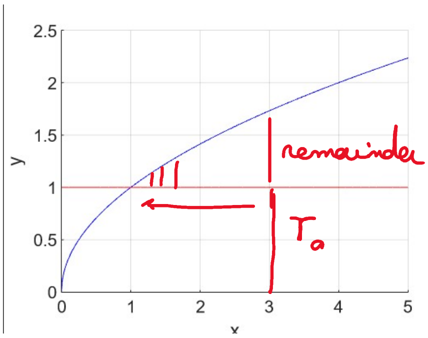
Approximate it with a Taylor polynomial of degree 2.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots$$

Taylor pol[illegible] of degree 0 :

Taylor polynomial of degree 0:

$$T_0(x_0) = f(x_0) = f(1) = \sqrt{1} = 1$$



that goes to 0 when $x \to x_0$

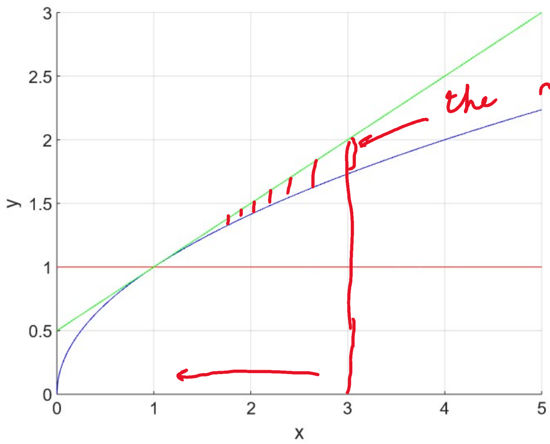
$$T_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \qquad D\, x^{\alpha} = \alpha\, x^{\alpha - 1}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(x_0) = f'(1) = \frac{1}{2}$$

$$T_1(x) = 1 + \frac{1}{2}(x - 1) = 1 + \frac{1}{2}x - \frac{1}{2} = \frac{1}{2} + \frac{1}{2}x$$



the remainder tends to 0 as $x \to 1$

$$T_2(x) = \underline{f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2}$$

$$f''(x) = D\,\frac{1}{2\sqrt{x}} = \frac{1}{2}\,D\,x^{-\frac{1}{2}} =$$

$$= \frac{1}{2}\left(-\frac{1}{2}\right) x^{-\frac{1}{2}-1} = -\frac{1}{4}\,x^{-\frac{3}{2}} =$$

$$= -\frac{1}{4}\,\frac{1}{x\sqrt{x}}$$

$$f''(x_0) = f''(1) = -\frac{1}{4}$$

$$T_2(x) = \frac{1}{2} + \frac{1}{2}x - \frac{1}{8}(x-1)^2$$

$$f(x) = \underline{f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2} + $$
$$+ R_2(x, x_0)$$

$$f(\underline{x}) = f(\underline{x}_0) + \nabla f^T(\underline{x}_0)(\underline{x} - \underline{x}_0) +$$

$$+\ \frac{1}{2}(\underline{x}-\underline{x}_0)^T \underline{\underline{H}}f(\underline{x}_0)(\underline{x}-\underline{x}_0) + R_2(\underline{x},\underline{x}_0)$$

$$f(x,y) = x\sqrt{y} \qquad P_0(1,4)$$

$$T_0(x,y) = f(\underline{x}_0) = f(1,4) = 1\cdot\sqrt{4} = 2$$

$$T_1(x,y) = f(\underline{x}_0) + \nabla f^T(\underline{x}_0)(\underline{x}-\underline{x}_0)$$

$$f(x,y) = x\sqrt{y}$$

$$f'_x = \sqrt{y} \qquad f'_y = \frac{x}{2\sqrt{y}}$$

$\underline{x}_0$: particular point $(1,4)$

$\underline{x}$: generic point

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\nabla f(x,y) = \begin{bmatrix} \sqrt{y} \\ \dfrac{x}{2\sqrt{y}} \end{bmatrix}$$

$$\nabla f(\underline{x}) = \nabla f(x,y) = \begin{bmatrix} \sqrt{y} \\ \dfrac{x}{2\sqrt{y}} \end{bmatrix}$$

$$\nabla f(\underline{x}_0) = \nabla f(1,4) = \begin{bmatrix} \sqrt{4} \\ \dfrac{1}{2\sqrt{4}} \end{bmatrix} = \begin{bmatrix} 2 \\ \dfrac{1}{4} \end{bmatrix}$$

$$\underline{x} - \underline{x}_0 = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} x-1 \\ y-4 \end{bmatrix}$$

$$T_1(x,y) = f(\underline{x}_0) + \nabla f^T(\underline{x}_0)(\underline{x} - \underline{x}_0) =$$

$$= 2 + \begin{bmatrix} 2 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x-1 \\ y-4 \end{bmatrix} =$$

$$= 2 + 2(x-1) + \frac{1}{4}(y-4) =$$

$$= 2 + 2x - 2 + \frac{1}{4}y - 1 = -1 + 2x + \frac{1}{4}y$$

$$f'_x = \sqrt{y} \qquad f'_y = \frac{x}{2\sqrt{y}}$$

$$f''_{xx} = 0 \qquad f''_{xy} = \frac{\partial}{\partial y}\sqrt{y} = \frac{1}{2\sqrt{y}}$$

$$f''_{yx} = \frac{\partial}{\partial x}\frac{x}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}$$

$$f''_{yy} = \frac{\partial}{\partial y}\frac{x}{2\sqrt{y}} = \frac{x}{2}\frac{\partial}{\partial y} y^{-\frac{1}{2}} =$$

$$= \frac{x}{2}\left(-\frac{1}{2}\right) y^{-\frac{3}{2}} = -\frac{x}{4y\sqrt{y}}$$

$$\underline{\underline{H}}f(\underline{x}) = \underline{\underline{H}}f(x,y) = \begin{bmatrix} 0 & \frac{1}{2\sqrt{y}} \\ \frac{1}{2\sqrt{y}} & -\frac{x}{4y\sqrt{y}} \end{bmatrix}$$

$$\underline{\underline{H}}f(\underline{x}_0) = \underline{\underline{H}}f(1,4) = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{32} \end{bmatrix}$$

$$(\underline{x} - \underline{x}_0)^T \underline{\underline{H}}f^{(\underline{x}_0)} (\underline{x} - \underline{x}_0)$$

$$\begin{bmatrix} x-1 & y-4 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{32} \end{bmatrix} \begin{bmatrix} x-1 \\ y-4 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{y-4}{4} & \frac{x-1}{4} - \frac{y-4}{32} \end{bmatrix} \begin{bmatrix} x-1 \\ y-4 \end{bmatrix} =$$

$$= \frac{1}{4}(x-1)(y-4) + \frac{(x-1)(y-4)}{4} - \frac{(y-4)^2}{32} =$$

$$= \frac{1}{2}(x-1)(y-4) - \frac{(y-4)^2}{32}$$

$$T_2(x,y) = -1 + 2x + \frac{1}{4}y + \frac{1}{4}(x-1)(y-4) - \frac{1}{64}(y-4)^2$$