TUSDAY, April 30, 4 pm - 6 pm

$f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ $\qquad$ A open

$$f(\underline{x}) = f(\underline{x}_0) + \nabla f(\underline{x}_0)^T (\underline{x} - \underline{x}_0) +$$
$$+ \frac{1}{2}(\underline{x} - \underline{x}_0)^T \underline{\underline{H}}f(\underline{x}_0)(\underline{x} - \underline{x}_0) + R_2(\underline{x}, \underline{x}_0)$$

Suppose that $\underline{x}_0$ is a critical point, then this formula becomes:

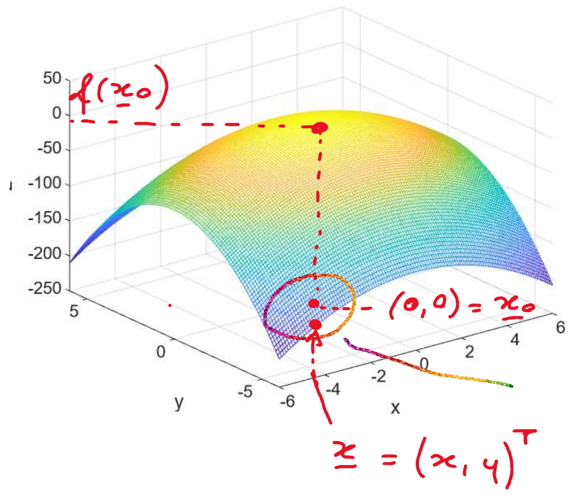
$$f(\underline{x}) = f(\underline{x}_0) + \cancel{\nabla f(\underline{x}_0)^T (\underline{x} - \underline{x}_0)}^{\,0} +$$
$$+ \frac{1}{2}(\underline{x} - \underline{x}_0)^T \underline{\underline{H}}f(\underline{x}_0)(\underline{x} - \underline{x}_0) + \underbrace{R_2(\underline{x}, \underline{x}_0)}_{\approx 0}$$

in a neighborhood of $\underline{x}_0$

$$f(\underline{x}) - f(\underline{x}_0) \approx \frac{1}{2}(\underbrace{\underline{x} - \underline{x}_0}) ^T \underline{\underline{H}}f(\underline{x}_0)(\underbrace{\underline{x} - \underline{x}_0})$$

Let me call this vector $\underline{h}$, where $\underline{h} = \underline{x} - \underline{x}_0$

$$f(\underline{x}) = f(x, y) = 6 - 2x^2 - 4y^2$$



neighborhood of $\underline{x}_0$

$$f(\underline{x}) - f(\underline{x}_0) \approx \frac{1}{2} \underbrace{\underline{h}^T \underline{\underline{H}}f(\underline{x}_0)\, \underline{h}}$$

If $\underline{\underline{H}}f(\underline{x}_0)$ is positive definite, then

$\underline{h}^T \underline{\underline{H}}f(\underline{x}_0)\, \underline{h} > 0$ then

$f(\underline{x}) - f(\underline{x}_0) > 0$ $\quad \forall\ \underline{x}$ in a neighborhood of $\underline{x}_0$

then $f(\underline{x}) > f(\underline{x}_0)$ $\quad \forall\ \underline{x}$ in a neighborhood of $\underline{x}_0$

then $\underline{x}_0$ is a local minimum.

If $\underline{\underline{H}}f(\underline{x}_0)$ is negative definite, then

$\underline{h}^T \underline{\underline{H}}f(\underline{x}_0)\, \underline{h} < 0$ then

$f(\underline{x}) - f(\underline{x}_0) < 0$ $\quad \forall\ \underline{x}$ in a neighborhood of $\underline{x}_0$

neighborhood of $\underline{x}_0$

then $f(\underline{x}) < f(\underline{x}_0)$ $\forall$ $\underline{x}$ in a neighborhood of $\underline{x}_0$

then $\underline{x}_0$ is a local maximum.

If $\underline{\underline{H}}f(\underline{x}_0)$ is indefinite, $\underline{x}_0$ is a saddle point.

This is an informal proof

Second order condition: THEOREM

Let $f: A \subseteq R^n \to R$ be a C^2 function and $\underline{x}^* \in A$ is an interior cirtical point of A.

(1) If the Hessian Hf($\underline{x}^*$) is a negative definite matrix then $\underline{x}^*$ is a relative MAX of f

(2) If the Hessian Hf($\underline{x}^*$) is a positive definite matrix then $\underline{x}^*$ is a relative MIN of f

(3) If the Hessian Hf($\underline{x}^*$) is indefinite then $\underline{x}^*$ is neither a relative MAX nor a relative MIN of f. It is a SADDLE POINT.

Notice that: the previous Theorem states only a sufficient condition!

In fact, if the Hessian matrix is **semi-definite** in an interior critical point, then nothing can be said about the nature of that critical point!

We will solve analytically some problems of Unconstrained Optimization that are not TOO COMPLEX.

$$\underline{\underline{H}}f(\underline{x}_0) = \begin{vmatrix} f''_{xx}(\underline{x}_0) & f''_{xy}(\underline{x}_0) \\ f''_{xy}(\underline{x}_0) & f''_{yy}(\underline{x}_0) \end{vmatrix}$$

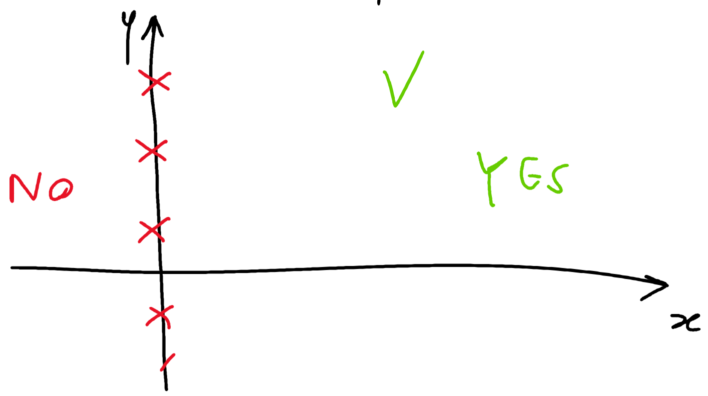
$f''_{xx}(\underline{x}_0) > 0$, positive definit.

$$f''_{xx}(\underline{x}_0) > 0 \qquad \text{; positive definit}$$

$$\det \underline{\underline{H}}_f(\underline{x}_0) > 0 \quad \longleftrightarrow \quad \underline{\underline{H}}_f(\underline{x}_0) \succ 0$$

Ex9: Determine the local max and min points of the following function: $z = \ln x - 2x^2 + y^4 - 32y$

$$z = f(x,y) = \ln x - 2x^2 + y^4 - 32y$$

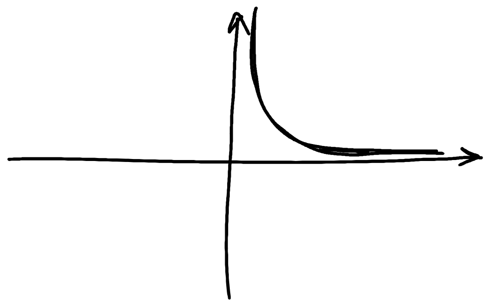


$$\operatorname{dom} f = \{(x,y) \in \mathbb{R}^2 : x > 0\}$$

$$z = f(x,y) = \ln x - 2x^2 + y^4 - 32y$$

$$f \in C$$

$$f'_x = \frac{1}{x} - 4x \quad \text{is continuous}$$



$$f'_y = 4y^3 - 32 \quad \text{is continuous}$$

$$f \in C^1$$

$$f''_{xx} = -\frac{1}{x^2} - 4 \quad \text{is } C$$

$f''_{xy} = 0 \qquad f''_{yx} = 0$ are all C

$f''_{yy} = 12y^2$

$f \in C^2$.

Critical points: $\nabla f(x, y) = \underline{0}$

$$\begin{cases} \frac{1}{x} - 4x = 0 \\ 4y^3 - 32 = 0 \end{cases} \qquad \begin{cases} 1 - 4x^2 = 0 \\ y^3 - 8 = 0 \end{cases} \qquad \begin{cases} x = \pm\frac{1}{2} \\ y = 2 \end{cases}$$

$\cancel{A\left(-\frac{1}{2}, 2\right)}$ and $B\left(\frac{1}{2}, 2\right)$

we discard it because it must be $x > 0$

$$\underline{\underline{H}}f(\underline{x}) = \underline{\underline{H}}f(x, y) = \begin{bmatrix} -\frac{1}{x^2} - 4 & 0 \\ 0 & 12y^2 \end{bmatrix}$$

$$\underline{\underline{H}}f(B) = \begin{bmatrix} -8 & 0 \\ 0 & 48 \end{bmatrix}$$

$\lambda_1 = -8$

$\lambda_2 = 48$ $\qquad \underline{\underline{H}}f(B)$ is indefinite

so B is a saddle point.

or

with the leading principal minor:

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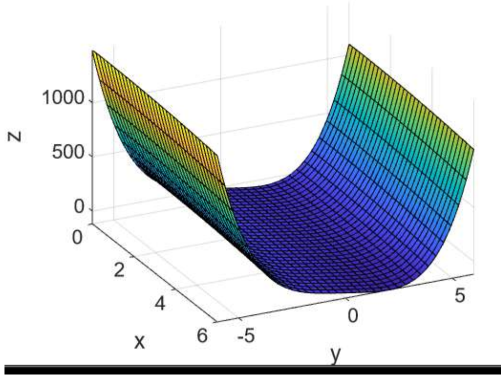
$|H_1| = -8 < 0$

$|H_2| = -8 \cdot 48 - 0 \cdot 0 < 0$ indefinite.



8. $f(x, y) = x^2y^3$, $\mathbf{x}_0 = (1, 1)$

$$f(\underline{x}) = f(\underline{x}_0) + \nabla f(\underline{x}_0)^T(\underline{x} - \underline{x}_0) + \frac{1}{2}(\underline{x} - \underline{x}_0)^T \underline{\underline{H}}_f(\underline{x}_0)(\underline{x} - \underline{x}_0) +$$
$$+ R_2(\underline{x}, \underline{x}_0)$$

$$T_0(x, y) = f(\underline{x}_0) = f(1, 1) = 1^2 \, 1^3 = 1$$

$$T_1(x, y) = f(\underline{x}_0) + \nabla f(\underline{x}_0)^T(\underline{x} - \underline{x}_0)$$

$f'_x = 2xy^3$ $\quad f'_y = 3x^2y^2$

$$\nabla f(\underline{x}_0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$T_1(x, y) = 1 + \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 1 \end{bmatrix} =$$

$$= 1 + 2(x-1) + 3(y-1)$$

$$f''_{xx} = 2y^3 \qquad f''_{xy} = 6xy^2 = f''_{yx} \qquad f''_{yy} = 6x^2y$$

$$\underline{\underline{H}}f(1,1) = \begin{bmatrix} 2 & 6 \\ 6 & 6 \end{bmatrix}$$

$$\frac{1}{2}(\underline{x} - \underline{x}_0)^T \underline{\underline{H}}f(1,1)(\underline{x} - \underline{x}_0) =$$

$$= \frac{1}{2} \begin{bmatrix} x-1 & y-1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} x-1 & y-1 \end{bmatrix} \begin{bmatrix} 2(x-1) + 6(y-1) \\ 6(x-1) + 6(y-1) \end{bmatrix} =$$

$$= \frac{1}{2} \left[2(x-1)^2 + 6(x-1)(y-1) + 6(x-1)(y-1) + 6(y-1)^2 \right] =$$

$$= (x-1)^2 + 6(x-1)(y-1) + 3(y-1)^2$$

$$T_2(x,y) = 1 + 2(x-1) + 3(y-1) +$$
$$+ (x-1)^2 + 6(x-1)(y-1) + 3(y-1)^2$$