$$f : \underset{\text{open}}{A} \subseteq \mathbb{R}^n \to \mathbb{R} \qquad \in C^1(A)$$

$\nabla f(\underline{x}^*) = \underline{0}$ $\qquad$ $\underline{x}^*$ critical point

$\nabla f(\underline{\hat{x}}) \neq \underline{0}$ $\qquad$ $\underline{\hat{x}}$ regular point

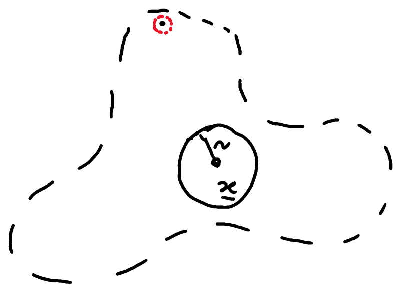
(it means that the gradient of f at $\underline{\hat{x}}$ is $\neq \underline{0}$)

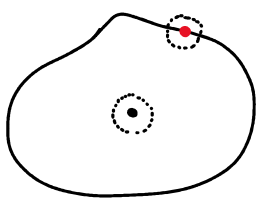
for all $\qquad$ there exists

A is open if $\forall\ \underline{x} \in A \quad \exists\ r > 0$
such that $B_r(\underline{x}) \subseteq A$

or

$B(\underline{x}, r)$





closed set

CLOSED SETS

A set $S \subseteq \mathbb{R}^m$ is closed if, whenever $\{\underline{x}_m\}_{m=1}^{+\infty}$ is a convergent sequence completely contained in S, its limit is also contained in S.

green

$$x_m = 1 + \frac{1}{m}$$

tends to 0

$$y_m = \frac{1}{m+1}$$

$m = 1$ $\quad x_1 = 1 + \frac{1}{1} = 2$ $\quad y_1 = \frac{1}{2}$

$m = 2$ $\quad x_2 = 1 + \frac{1}{2} = \frac{3}{2}$ $\quad y_2 = \frac{1}{3}$

$\vdots$

$x_3 = 1 + \frac{1}{3} = \frac{4}{3}$

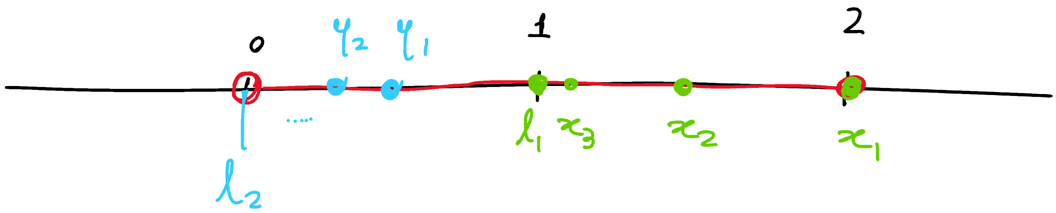
$\vdots$

$m = 100$ $\quad x_{100} = 1 + \frac{1}{100}$ $\quad y_{100} = \frac{1}{101}$

$$\lim_{m \to +\infty} x_m = 1 = l_1$$

$$\lim_{m \to +\infty} y_m = 0 = l_2$$

$$A = (0, 2]$$



x_m is completely contained in A and its

x_n is completely contained in A and its limit is in A

y_n is completely contained in A but its limit is not in A

Consequently A is not a closed set.

$B = [0 \quad 2]$ is a closed set.

Theorem: A set A is open if and only if its complementary is a closed set.

Complementary set:

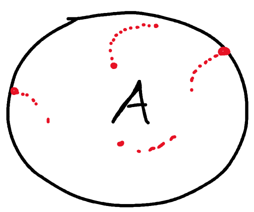
$A = \{$ Adrien, Eli, David, Zümrüt, Godwin $\}$

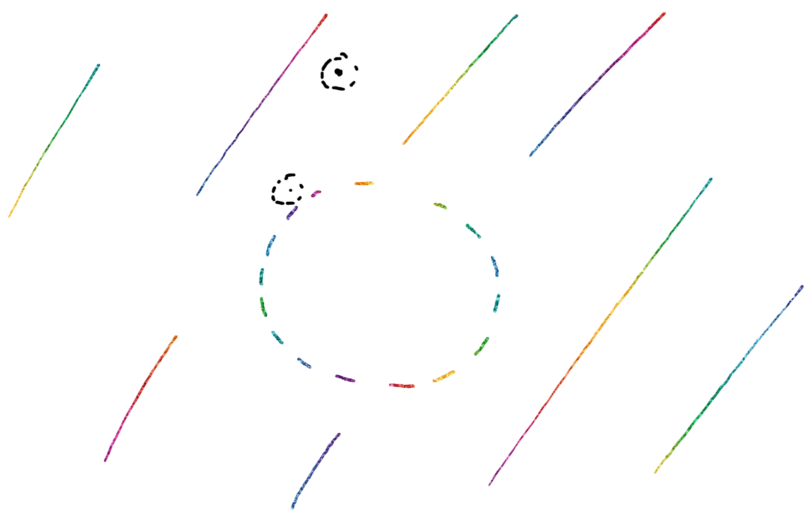
Universe set: the set of the students attending the IFC course, i.e., the biggest set of the context.

The complementary of A is all that is not A

$\bar{A}$ or A^c

$$\bar{A} = U - A \quad \text{or} \quad U \setminus A$$

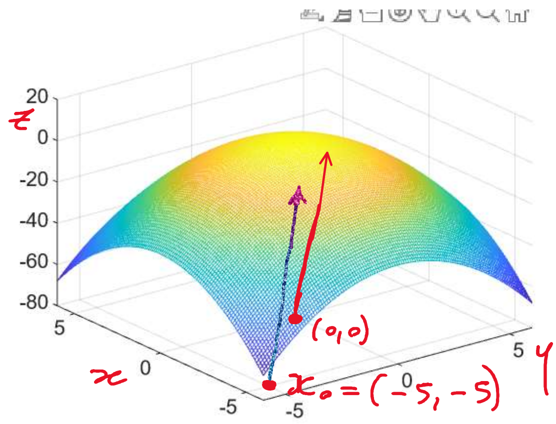




$\bar{A}$ is open

THEOREM 1: let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. At any regular point $\underline{x}$ inside A at which $\nabla f(\underline{x}) \neq \underline{0}$, the gradient $\nabla f(\underline{x})$ points at $\underline{x}$ into the direction in which f increases most rapidly.

```
>> X = -6:.1:6;
>> Y = X;
>> [x, y] = meshgrid(X, Y);
>> z = 4 - x.^2 - y.^2;
>> mesh(x, y, z)
```



$$f(x,y) = 4 - x^2 - y^2$$

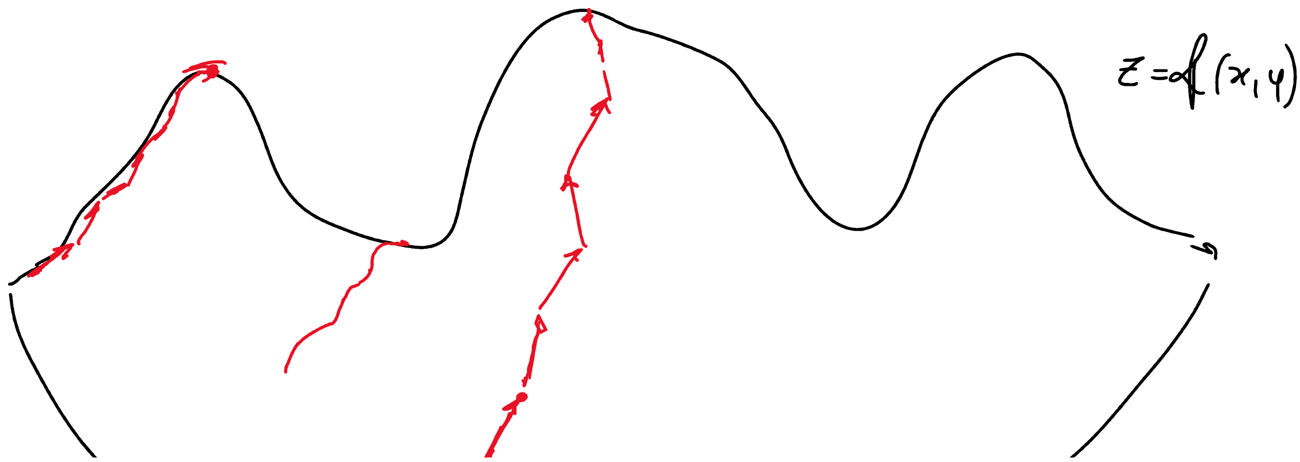
$$f'_x = -2x$$

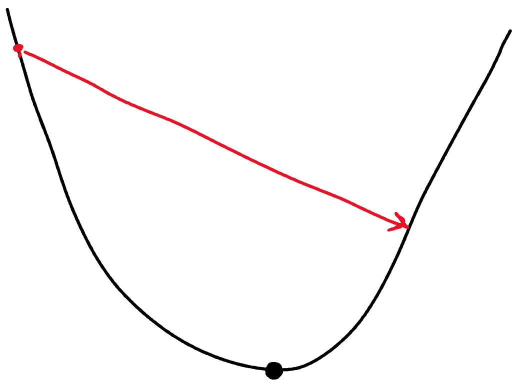
$$f'_y = -2y$$

$$\nabla f(x,y) = (-2x, -2y)^T$$

$$\nabla f(\underline{x}_0) = \nabla f(-5,-5) =$$

$$= (10, 10)$$



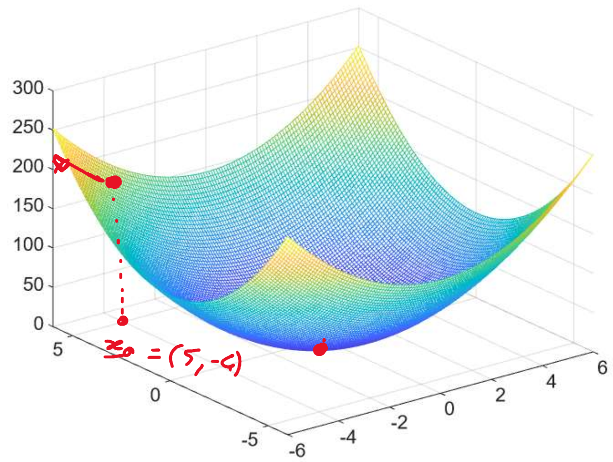


```
>> z = 3*x.^2 + 4*y.^2;
>> mesh(x, y, z)
```

$$z = 3x_1^2 + 4x_2^2$$

$$f(\underline{x}).$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$



f = @(x) 3*x(1)^2 + 4*x(2)^2

f =

function_handle with value:

@(x)3*x(1)^2+4*x(2)^2

x0 = [5, -4];
[xMin, fMin] = fminsearch(f, x0)

xMin =

1.0e-04 *

-4

xMin =

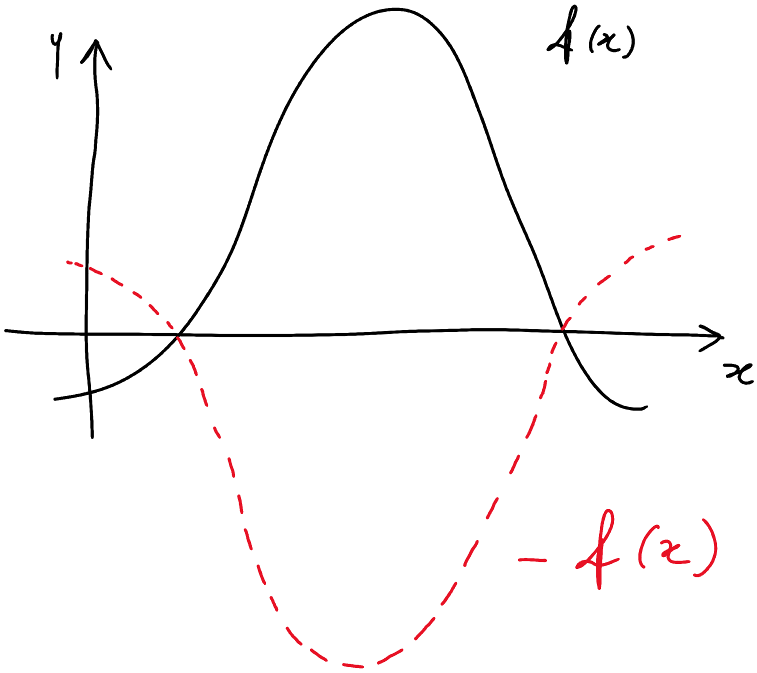
1.0e-04 *

0.2135   -0.4512

$0.2135 \cdot 10^{-4} \approx 0$

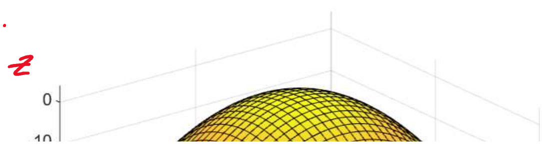
fMin =

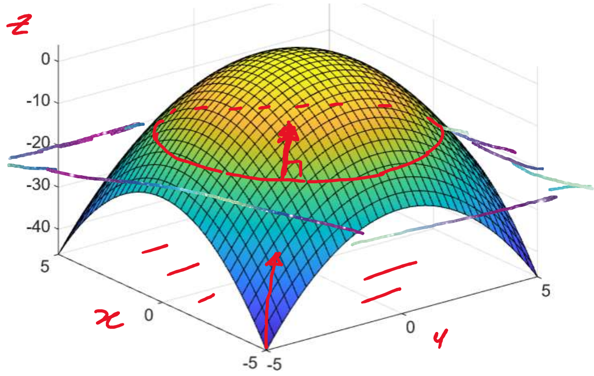
9.5104e-09 $\simeq 9.5 \cdot 10^{-9} \approx 0$



THEOREM 2: let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and let $\underline{x}_0$ be an interior point of A such that $\nabla f(\underline{x}_0) \neq \underline{0}$. Then the gradient vector $\nabla f(\underline{x}_0)$ is perpendicular to the level set $f(x_1, x_2, \dots, x_n) = z$ at $\underline{x}_0$

fsurf(@(x, y) 4 - x.^2 - y.^2)

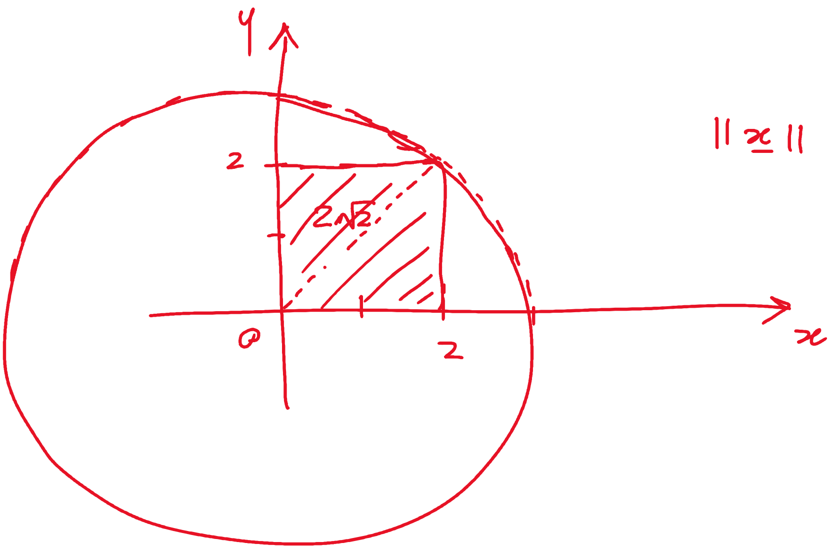




Bounded sets:

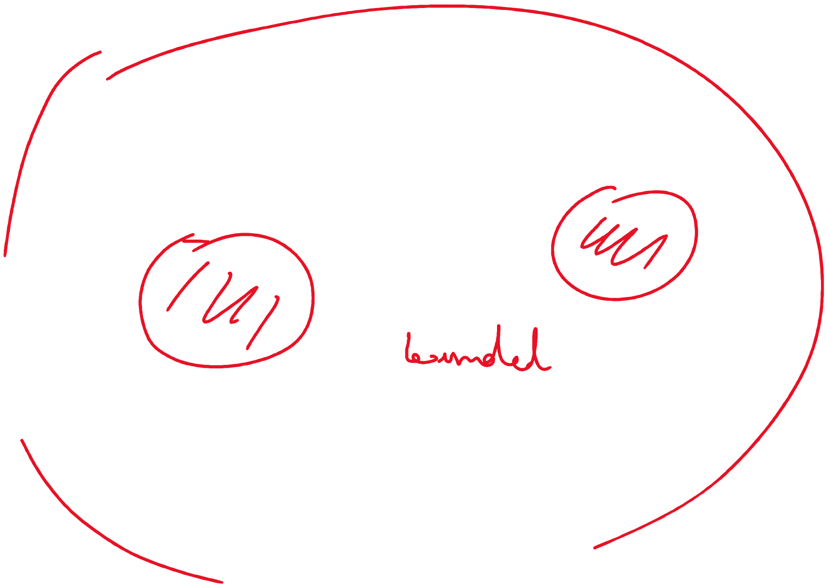
A set $A \subseteq \mathbb{R}^n$ is bounded if there exists a number B such that $\|\underline{x}\| \leq B \quad \forall\ \underline{x} \in A$.

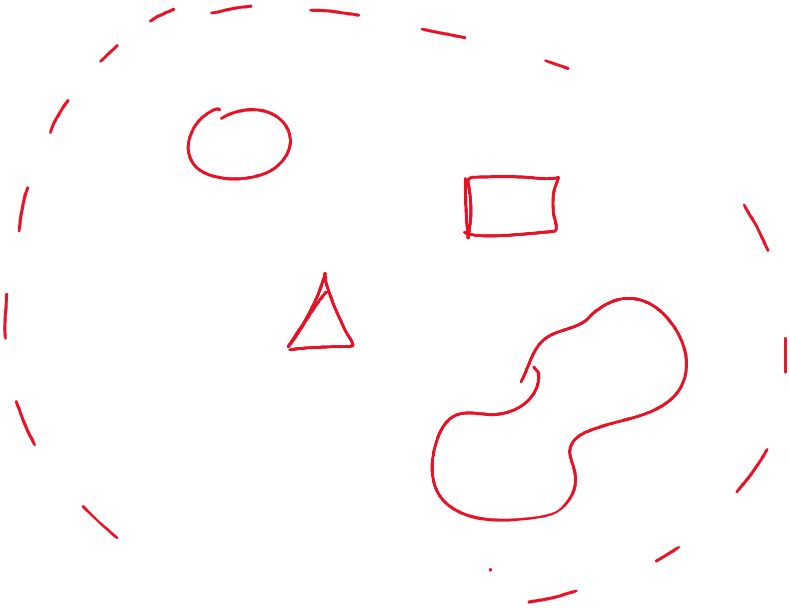
That is, if A is contained in some ball.

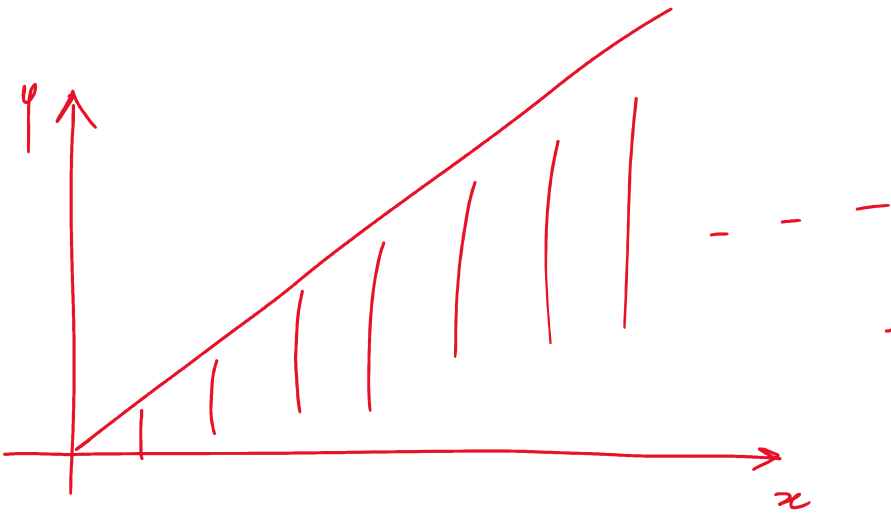


$\|\underline{x}\| \leq 2\sqrt{2}$

bounded







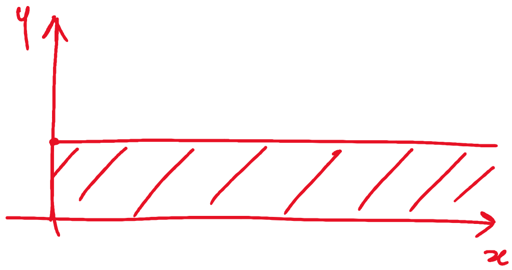
unbounded

COMPACT SETS

A set $A \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.

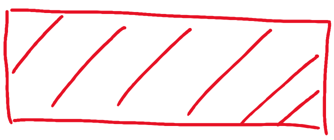
bounded
open

not compact



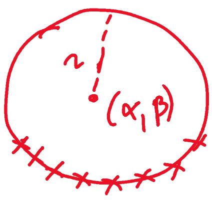
closed
unbounded

not compact



closed
bounded

compact.



$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$

bounded
closed $\Big\}$ compact.

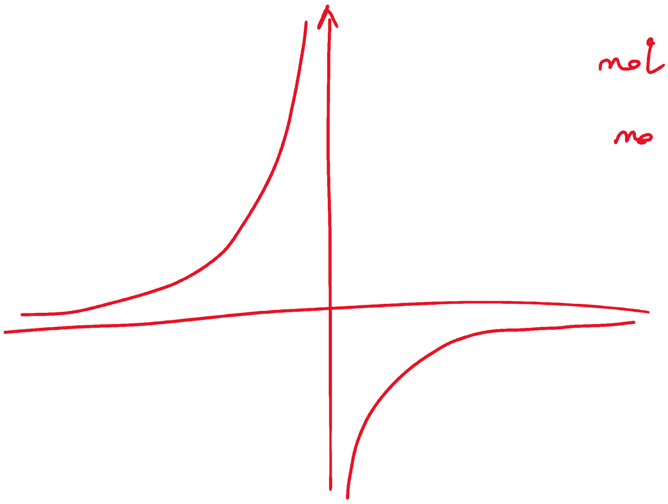
its complementary is open

WEIERSTRASS'S THEOREM (WEIERSTRAß)

Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function whose domain A is a compact set. Then, there exist points $\underline{x}_m$ and $\underline{x}_M$ in A such that:

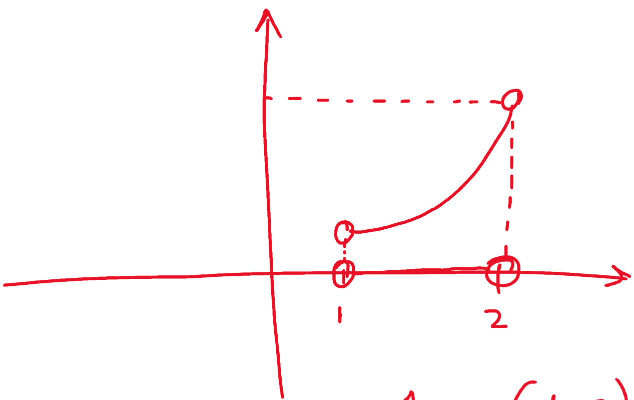
$$f(\underline{x}_m) \leq f(\underline{x}) \leq f(\underline{x}_M) \quad \forall \, \underline{x} \in A$$

That is, $\underline{x}_m$ is a global minimum and $\underline{x}_M$ is a global maximum.

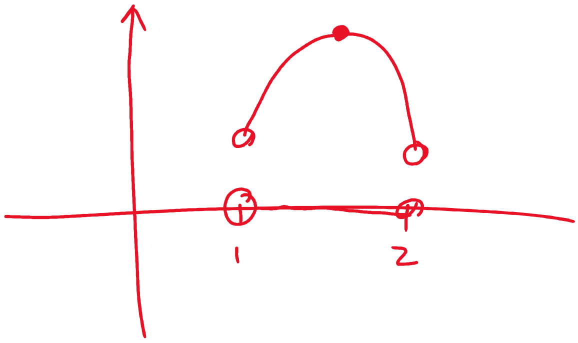


not continuous

no global maximum/minimum



$A = (1,2)$


1
2