CONSTRAINED OPTIMIZATION

$$\max \quad f(x_1, x_2, \dots, x_n)$$

$$\text{s.t.:} \quad \left.\begin{aligned} g_1(x_1, x_2, \dots, x_n) &\leq b_1 \\ &\vdots \\ g_k(x_1, x_2, \dots, x_n) &\leq b_k \end{aligned}\right\} \begin{array}{c} k \\ \text{inequality} \\ \text{constraints} \end{array}$$

$$\left.\begin{aligned} h_1(x_1, x_2, \dots, x_n) &= c_1 \\ &\vdots \\ h_m(x_1, x_2, \dots, x_n) &= c_m \end{aligned}\right\} \begin{array}{c} m \\ \text{equality} \\ \text{constraints} \end{array}$$

utility maximization problem

$$\max \quad U(x_1, x_2, \dots, x_n)$$

$$p_1 x_1 + p_2 x_2 + \dots + p_n x_n \leq B$$

$$x_i \geq 0 \qquad \forall \ i \in \{1, \dots, n\}$$

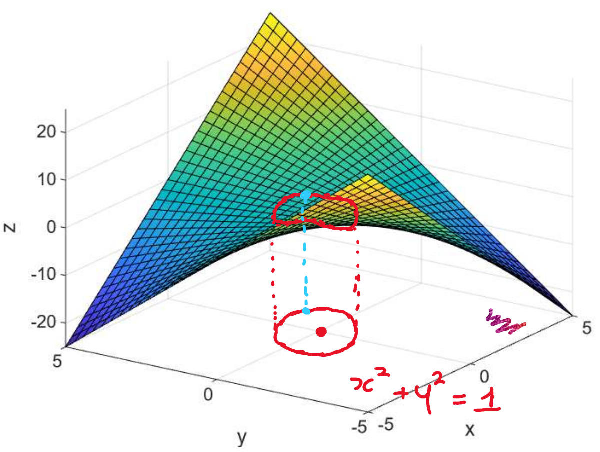
$$x_i \geq 0 \qquad -x_i \leq 0$$

$$\infty$$

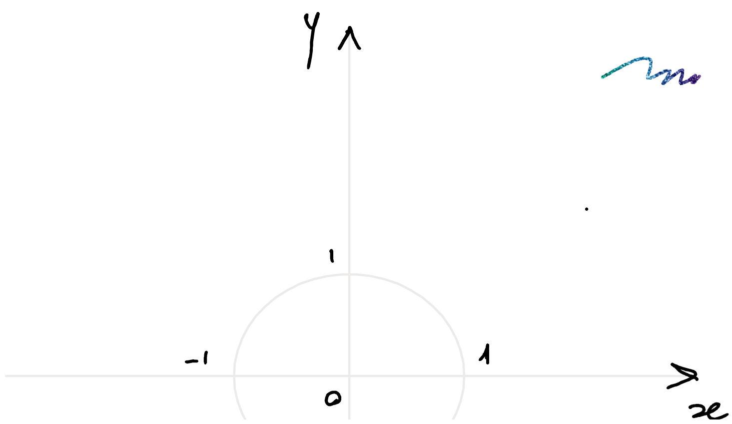
$$\max \quad f(x, y)$$
$$\text{s.t.:} \quad h(x, y) = c$$

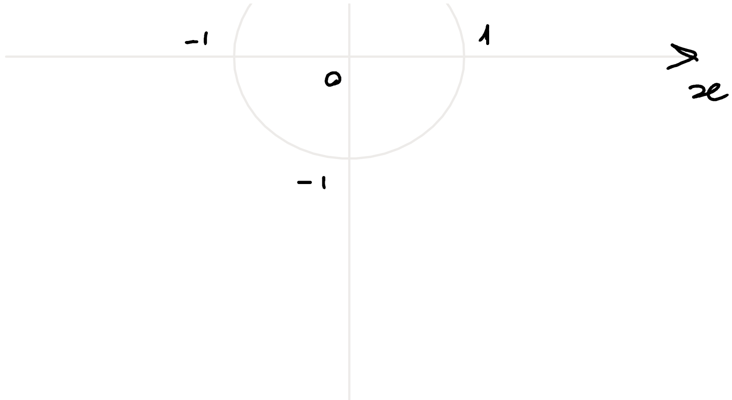
$$\max \quad z = xy \qquad \text{objective function}$$
$$\text{s.t.} \quad x^2 + y^2 = 1 \qquad \text{constraint}$$

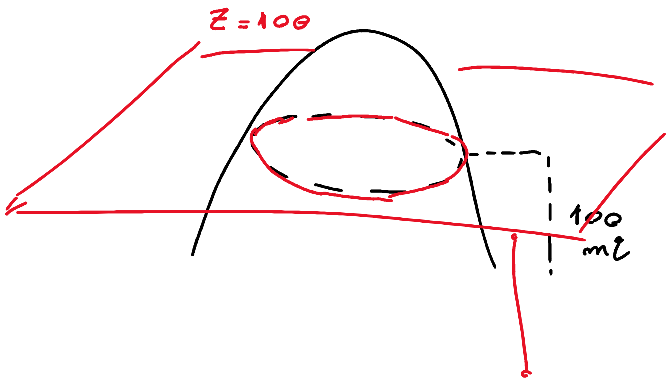
```
f = @(x, y) x.*y;
fsurf(f)
xlabel('x')
ylabel('y')
zlabel('z')
```

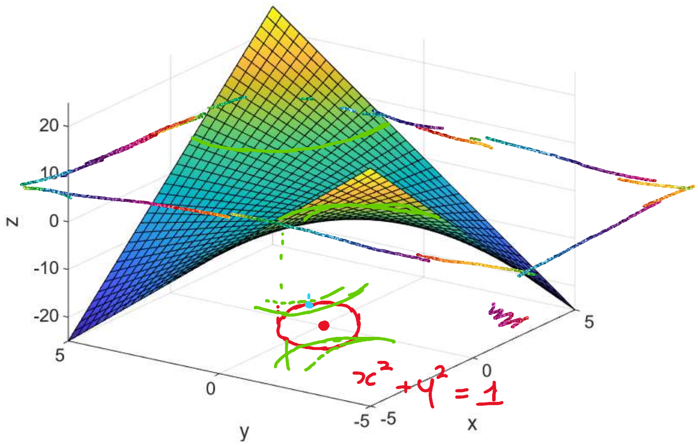


$$(x - \alpha)^2 + (y - \beta)^2 = r^2$$
$$(x - 0)^2 + (y - 0)^2 = 1^2$$









$z = k$

$$\begin{cases} z = f(x,y) \\ z = k \end{cases}$$

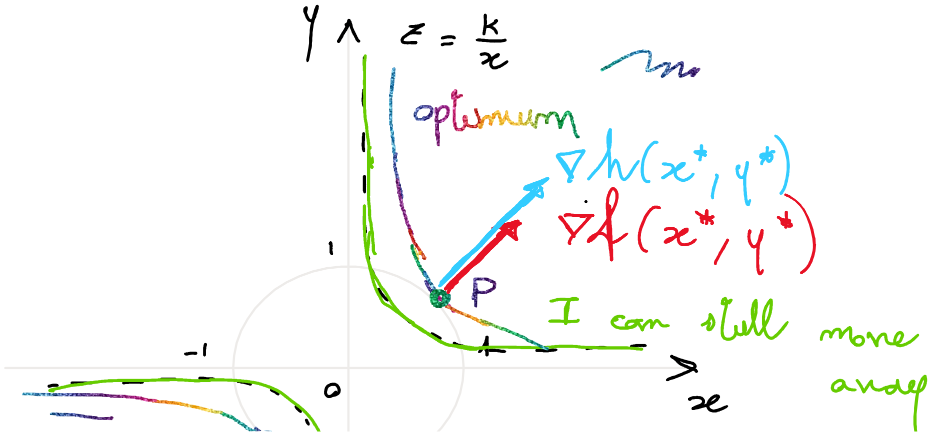
$$f(x,y) = k$$

$$xy = k$$

$$y = \frac{k}{x}$$

level curves

$f(x,y) = k$



$\underbrace{x^2 + y^2}$ $= 1$ it is a particular level curve with $k = 1$

$h(x, y) = x^2 + y^2$

Level curves of h:

$h(x, y) = k$

$\nabla h(x^*, y^*) \perp$ to the constraint set (i.e., the circle)

perpendicular

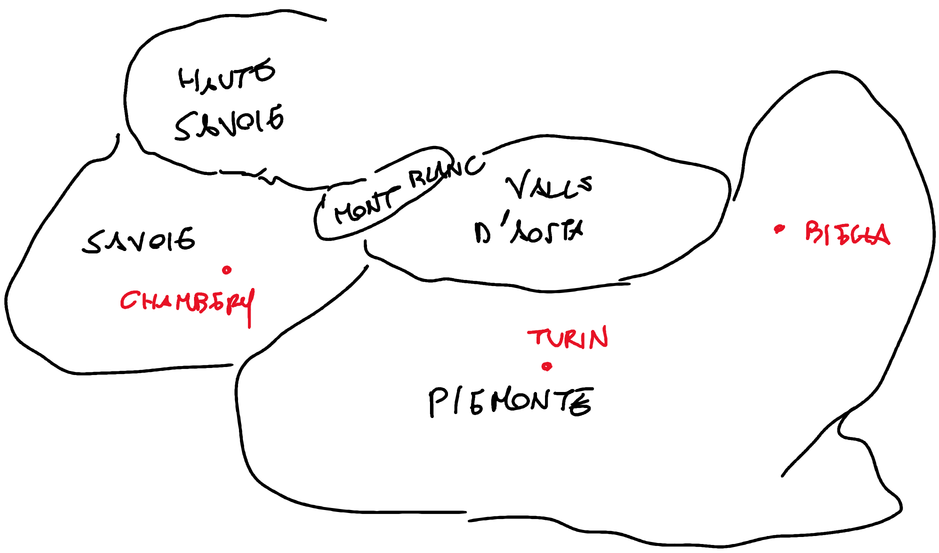
So $\nabla f(x^*, y^*)$ is parallel to $\nabla h(x^*, y^*)$

Thus $\exists \; \lambda$: $\boxed{\nabla f(x^*, y^*) = \lambda \nabla h(x^*, y^*)}$

λ is called Lagrange multiplier

Louis Lagrange

Luigi Lagrangia



$$\boxed{\nabla f(x^*, y^*) = \lambda \nabla h(x^*, y^*)}$$

$$\frac{\partial f}{\partial x}(x^*, y^*) = \lambda \frac{\partial h}{\partial x}(x^*, y^*) \qquad (1)$$

$$\frac{\partial f}{\partial y}(x^*, y^*) = \lambda \frac{\partial h}{\partial y}(x^*, y^*) \qquad (2)$$

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda\left(h(x, y) - c\right)$$

$$\nabla \mathcal{L} = \underline{0}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \qquad \frac{\partial f}{\partial x} - \lambda \frac{\partial h}{\partial x} = 0 \qquad \text{same as } (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \qquad \frac{\partial f}{\partial y} - \lambda \frac{\partial h}{\partial y} = 0 \qquad \text{same as } (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \qquad -\left(h(x,y) - c\right) = 0 \qquad \text{same as } h(x,y) = c$$

Theorem 18.1 Let f and h be C^1 functions of two variables. Suppose that $\mathbf{x}^* = (x_1^*, x_2^*)$ is a solution of the problem

$$\text{maximize} \quad f(x_1, x_2)$$
$$\text{subject to} \quad h(x_1, x_2) = c$$

Suppose further that (x_1^*, x_2^*) is not a critical point of h. Then, there is a real number μ^* such that (x_1^*, x_2^*, μ^*) is a critical point of the Lagrangian function

$$L(x_1, x_2, \mu) \equiv f(x_1, x_2) - \mu[h(x_1, x_2) - c].$$

In other words, at (x_1^*, x_2^*, μ^*)

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \text{and} \quad \frac{\partial L}{\partial \mu} = 0.$$

qualification constraints

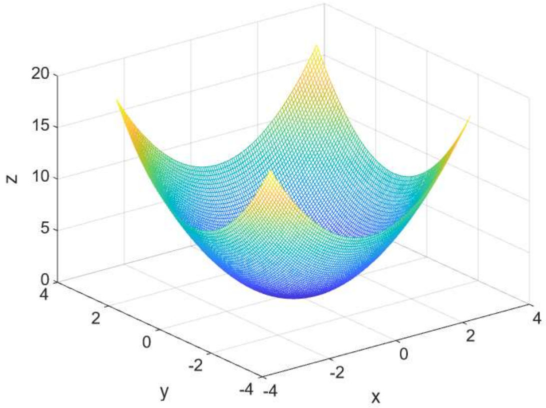
$$\nabla h(x^*, y^*) \neq \underline{0}$$

FIRST-ORDER (necessary) conditions

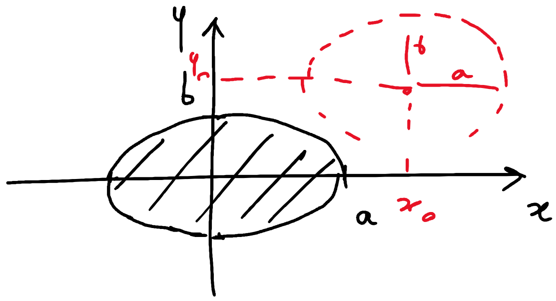
$$\max \quad f(x,y)$$
$$\text{s.t.:} \quad g(x,y) \leq b$$

$$\max \quad z = x^2 + y^2$$
$$\text{s.t.:} \quad \frac{x^2}{4} + y^2 \leq 1$$

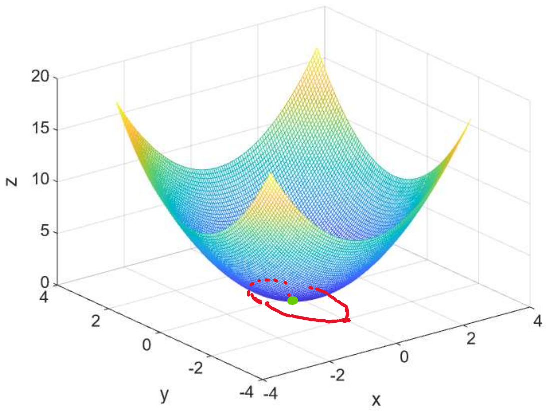
```
X = linspace(-3, 3, 100);
Y = X;
[x, y] = meshgrid(X, Y);
z = x.^2 + y.^2;
mesh(x, y, z)
xlabel('x')
ylabel('y')
zlabel('z')
```



$$\frac{x^2}{4} + y^2 = 1 \qquad\qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

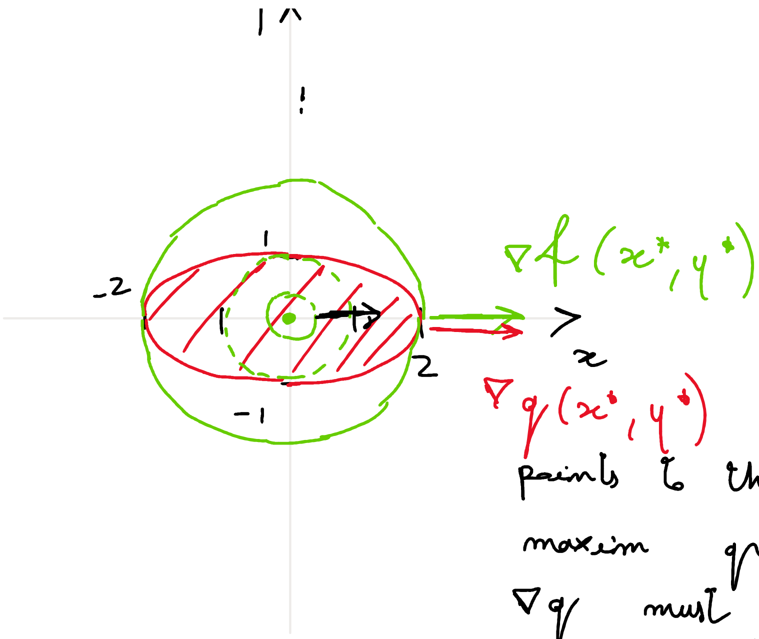


$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$



$$\begin{cases} z = x^2 + y^2 \\ z = k \end{cases}$$

$$x^2 + y^2 = k$$



points to the direction of maxim growth so ∇g must point outside the red domain and so it must point in the same direction of $\nabla f(x^*, y^*)$

$$g(x,y) = \frac{x^2}{4} + y^2 \leq 1$$

$$g(x,y) = k$$

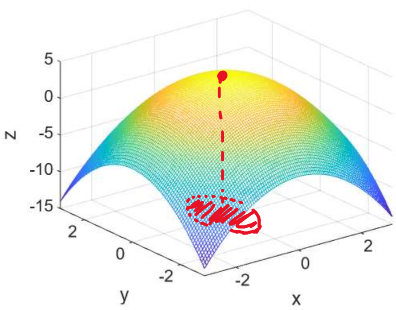
10 — 11

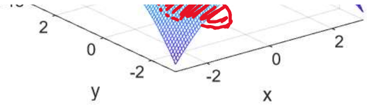
$$\boxed{\text{So } \lambda \geq 0}$$

$$\max \quad f(x,y) = 4 - x^2 - y^2$$

s.t. : $\quad \frac{x^2}{4} + y^2 \leq 1$

```
z = 4 - (x.^2 + y.^2);
mesh(x, y, z)
xlabel('x')
ylabel('y'), zlabel('z')
```





(B)

The other possibility is that our maximum is at an interior point of our domain. In this case $\nabla f(x^*, y^*) = 0$

That is: $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b)$

If I set $\lambda = 0$ then $\nabla \mathcal{L} = \nabla f$

(A) The maximum is on the border, that is:

$$g(x^*, y^*) = b$$

i.e. the inequality constraint is active or binding

So to summarize, either $\lambda = 0$ or $g(x^*, y^*) = b$

So:

$$\lambda \left[g(x^*, y^*) - b \right] = 0$$

complementary slackness

$(x - 1)(x - 2) = 0$ $\quad x - 1 = 0$ or $x - 2 = 0$

Theorem 18.3 Suppose that f and g are C^1 functions on $\mathbf{R}^2$ and that (x^*, y^*) maximizes f on the constraint set $g(x, y) \leq b$. If $g(x^*, y^*) = b$, suppose that

$$\frac{\partial g}{\partial x}(x^*, y^*) \neq 0 \quad \text{or} \quad \frac{\partial g}{\partial y}(x^*, y^*) \neq 0.$$

In any case, form the Lagrangian function

$$L(x, y, \lambda) = f(x, y) - \lambda \cdot [g(x, y) - b].$$

Then, there is a multiplier λ^* such that:

(a) $\frac{\partial L}{\partial x}(x^*, y^*, \lambda^*) = 0,$

(b) $\frac{\partial L}{\partial y}(x^*, y^*, \lambda^*) = 0,$

(c) $\lambda^* \cdot [g(x^*, y^*) - b] = 0,$

(d) $\lambda^* \geq 0,$

(e) $g(x^*, y^*) \leq b.$

∞

max $\quad Z = xy$

s.t. $\quad x^2 + y^2 = 1$

$h(x, y) = x^2 + y^2$

$\nabla h(x, y) \neq (0, 0)$

$(2x, 2y) \neq (0, 0)$

$(x, y) \neq (0, 0)$ it is not a problem because the origin does not lie on the constraint set.

$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda (h(x, y) - c) =$

$$= xy - \lambda\left(x^2 + y^2 - 1\right)$$

$$\nabla \mathcal{L}(x, y, \lambda) = \underline{0}$$

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2x\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - 2y\lambda = 0$$

$$\begin{cases} y - 2x\lambda = 0 \\ x - 2y\lambda = 0 \\ x^2 + y^2 = 1 \end{cases} \qquad \begin{cases} y = 2x\lambda \\ x - 4x\lambda^2 = 0 \\ x^2 + y^2 = 1 \end{cases}$$

$$x(1 - 4\lambda^2) = 0$$

1 $x = 0$ or 2 $1 - 4\lambda^2 = 0$

1

$$\begin{cases} y = 0 \\ x = 0 \\ 0 = 1 \end{cases} \quad \text{NO}$$

2

2.a $\lambda = \frac{1}{2}$ or 2.b $\lambda = -\frac{1}{2}$

2 . a

$$\begin{cases} y = x \\ \lambda = \frac{1}{2} \\ 2x^2 = 1 \end{cases} \qquad \begin{cases} y = \frac{1}{\sqrt{2}} \\ \lambda = \frac{1}{2} \\ x = \frac{1}{\sqrt{2}} \end{cases} \vee \begin{cases} y = -\frac{1}{\sqrt{2}} \\ \lambda = \frac{1}{2} \\ x = -\frac{1}{\sqrt{2}} \end{cases}$$

$$A\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \qquad B\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

candidates maximum points

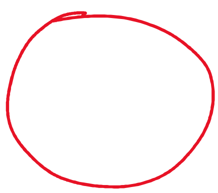
$$\begin{cases} y = -x \\ \lambda = -\frac{1}{2} \\ 2x^2 = 1 \end{cases}$$

$$C\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \qquad D\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\max \quad z = xy$$

$$\text{s.t.} \quad x^2 + y^2 = 1$$

continuous



bounded

closed

compact

So by Weierstrass's theorem there is a global maximum and a global minimum.

$$f(A) = f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$f(B) = \frac{1}{2}$$

$$f(C) = f(D) = -\frac{1}{2}$$

A and B are global maxima

C and D are global minima