pag. 417 example 18.4

$$\max \quad f(x_1, x_2) = x_1 x_2$$

s.t.: $\quad x_1 + 4x_2 = 16$

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(x_1 + 4x_2 - 16)$$

lambda

Constraint Qualification: $\nabla h(x_1, x_2) \neq (0,0)^T$

$$h(x_1, x_2) = x_1 + 4x_2 - 16$$

$$\frac{\partial h}{\partial x_1} = 1$$

$$\frac{\partial h}{\partial x_2} = 4$$

$$\nabla h(x_1, x_2) = (1, 4)^T \neq (0, 0)^T$$

so the constraint qualification is always satisfied.

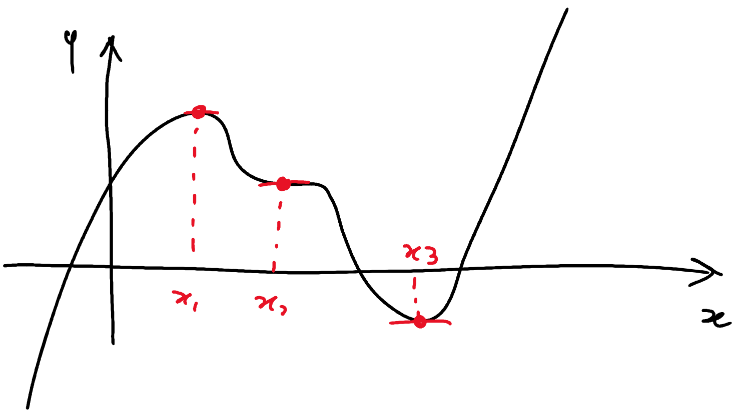
$$\mathcal{L}(x_1, x_2, \lambda) = x_1 x_2 - \lambda(x_1 + 4x_2 - 16)$$

$$\nabla \mathcal{L} = \underline{0}$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0, \quad \frac{\partial \mathcal{L}}{\partial x_2} = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

$$\begin{cases} x_2 - \lambda = 0 \\ x_1 - 4\lambda = 0 \\ x_1 + 4x_2 = 16 \end{cases} \qquad \begin{cases} x_2 = \lambda \\ x_1 = 4\lambda \\ 4\lambda + 4\lambda = 16 \end{cases} \qquad \begin{cases} x_2 = 2 \\ x_1 = 8 \\ \lambda = 2 \end{cases}$$

candidate point $P(8, 2)$



$f'(x) = 0$

you find x_1, x_2 and x_3

FIRST order (necessary) conditions.

second order (sufficient) conditions

if $f''(x^*) > 0$ then x^* local minimum

$<$ maximum

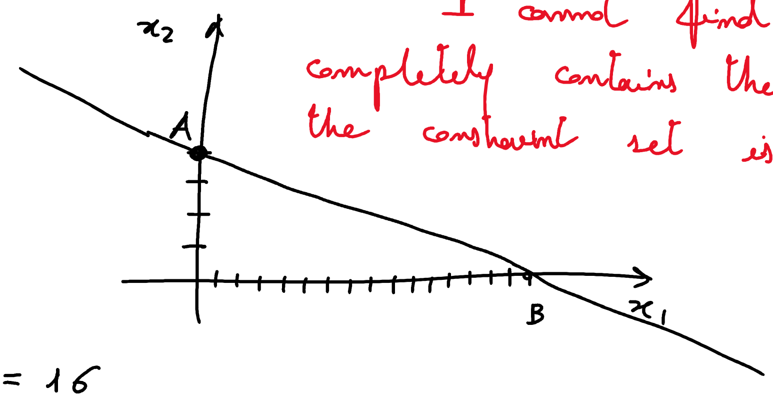
$$x_1 + 4x_2 = 16$$

$$ax + by + c = 0 \quad \text{line}$$

x_2

I cannot find any ball that

I cannot find any ball that completely contains the line, so the constraint set is unbounded



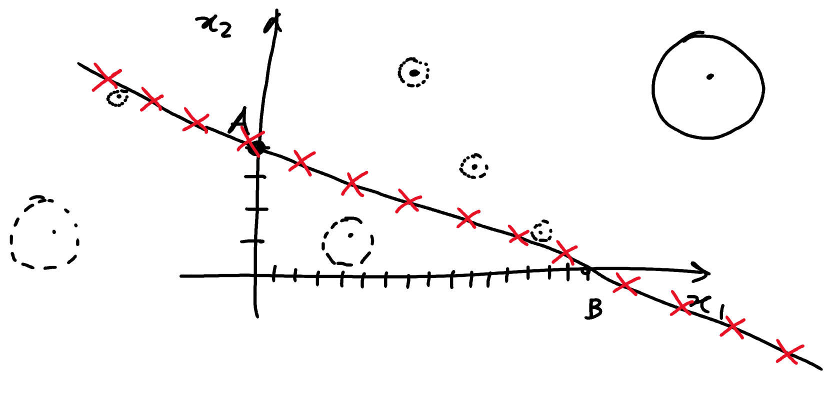
$x_1 + 4x_2 = 16$ $\qquad$ $x_1 + 4x_2 = 16$

If $x_1 = 0$ $\qquad$ $0 + 4x_2 = 16$ $\qquad$ $x_2 = 4$

$A(0,4)$

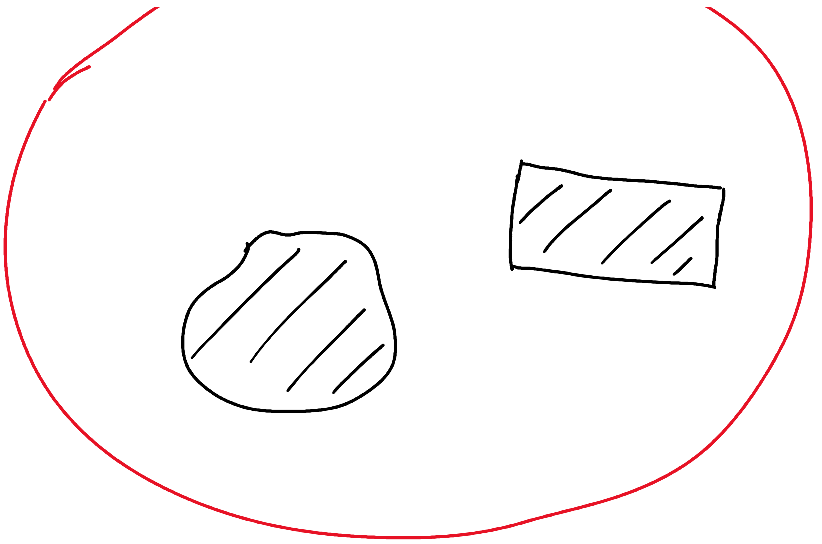
If $x_2 = 0$ $\qquad$ $x_1 + 0 = 16$ $\qquad$ $B(16,0)$

The complementary set of the constraint set (i.e., the line $x_1 + 4x_2 = 16$) is all the plane but the line



NO

The complementary set is opend and so the constaint set (i.e., the line) is closed.



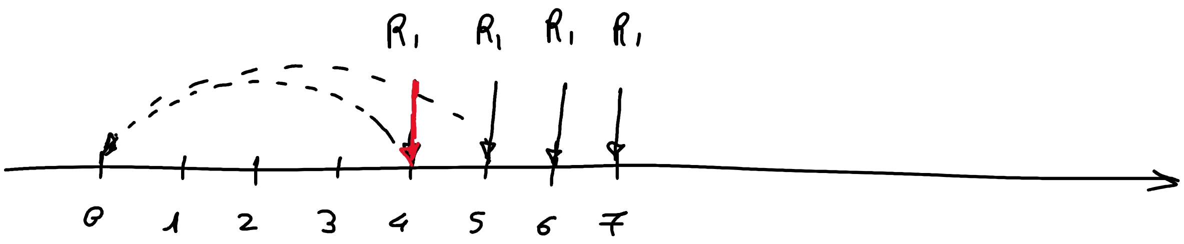
$= A$

is bounded

The constraint set is not compact and so we cannot apply Weierstrass' theorem.

1. Consider an annuity of equal and yearly payments of amount R_1 whose first payment occurs at year 4 and whose last payment occurs at year 7. Then, consider a perpetuity of equal and yearly payments of amount R_2 whose first payment occurs at year 10. Let the rate of (compound) interest be i. Let $R_1 = 1000$, $R_2 = 200$, and $i = 10\%$:

   - Find the value $w(\mathbf{R}_1, 0)$ of the annuity at year 0
   - Find the value $w(\mathbf{R}_2, 0)$ of the perpetuity at year 0
   - Find the value $w(0)$ of the annuity together with the perpetuity at year 0
   - Find the value $w(7)$ of the annuity together with the perpetuity at year 7



$$R_1 = w(0)(1+i)^4$$

$$w(0) = R_1(1+i)^{-4}$$

$$w(\underline{R}_1, 0) = R_1(1+i)^{-4} + R_1(1+i)^{\overset{-4-1}{-5}} + R_1(1+i)^{\overset{-4-2}{-6}} + R_1(1+i)^{\overset{-4-3}{-7}}$$

$$v := (1+i)^{-1}$$

$$w(\underline{R}_1, 0) = R_1(1+i)^{-4}\left(1 + (1+i)^{-1} + (1+i)^{-2} + (1+i)^{-3}\right) =$$

$$= R_1 v^4 \left(1 + v + v^2 + v^3\right) =$$

$$= R_1 v^4 \frac{1 - v^4}{1 - v}$$

$$S_m = 1 + v + v^2 + v^3 + \dots + v^m$$

$$S_m = 1 + v\left(1 + v + v^2 + \dots + v^{m-1}\right)$$

$$S_m = 1 + v\left(\underbrace{1 + v + v^2 + \dots + v^{m-1} + v^m}_{S_m} - v^m\right)$$

$$S_m = 1 + v\left(S_m - v^m\right)$$

$$S_m = 1 + r S_m - r^{m+1}$$

$$S_m - r S_m = 1 - r^{m+1}$$

$$(1-r) S_m = 1 - r^{m+1} \qquad r \neq 1$$

$$S_m = \frac{1 - r^{m+1}}{1 - r}$$

$$\lim_{m \to +\infty} S_m \qquad |r| < 1$$

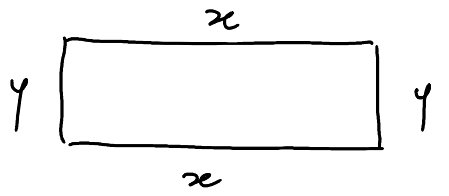
$$r = \frac{1}{2} \qquad \left(\frac{1}{2}\right)^{m+1} \xrightarrow{m \to +\infty} 0$$

$$\frac{1}{2} \quad \left(\frac{1}{2}\right)^2 = \frac{1}{4} \quad \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$\lim_{m \to +\infty} S_m = \frac{1}{1-r}$$

$$\infty$$

Find the rectangle of maximum area among those with perimeter equal to $2p$.



$$2x + 2y = 2p \quad \rightsquigarrow \quad x + y = p$$

$\max \quad xy$

s.t. $\quad x + y = p$

$f(x,y) = xy$

$h(x,y) = x + y - p$

Constraint qualification: $\nabla h(x^*, y^*) \neq \underline{0}$

$$\nabla h = (1, 1) \neq \underline{0} \qquad \checkmark$$

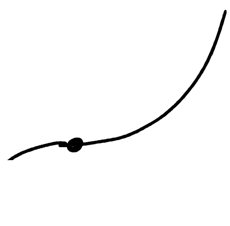
$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda\, h(x,y) =$$
$$= xy - \lambda(x + y - p)$$

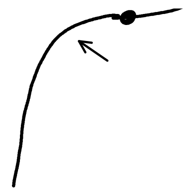
$\nabla \mathcal{L} = \underline{0}$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \qquad \frac{\partial \mathcal{L}}{\partial y} = 0 \qquad \begin{cases} y - \lambda = 0 \\ x - \lambda = 0 \\ x + y = p \end{cases} \qquad \begin{cases} y = \lambda \\ x = \lambda \\ 2\lambda = p \end{cases}$$

$$\begin{cases} y = \frac{p}{2} \\ x = \frac{p}{2} \\ \lambda = \frac{p}{2} \end{cases}$$

$\left(\frac{p}{2}, \frac{p}{2}\right)$ candidate maximum

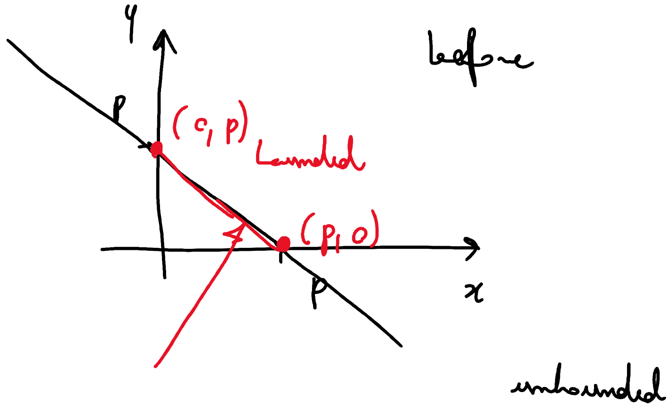




The constraint set $x+y=p$ is unbounded and so we cannot apply Weierstrass' theorem.

But, to be precise, the real problem is:

$$\max \quad xy$$
$$\text{s.t.} \quad x+y=p$$
$$x \geq 0$$
$$y \geq 0$$



Now the constraint set is bounded (and closed and so is compact). Consequently, we can apply Weierstrass' theorem. Thus, there is a global maximum and a global minimum

The global minimum is met at $(p,0)$ or $(0,p)$ that is a degenerate rectangle.

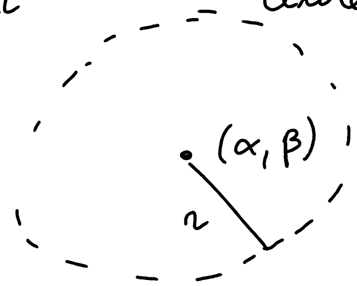
The global maximum, by exclusion, is $\left(\frac{p}{2}, \frac{p}{2}\right)$ that is a square.

∞

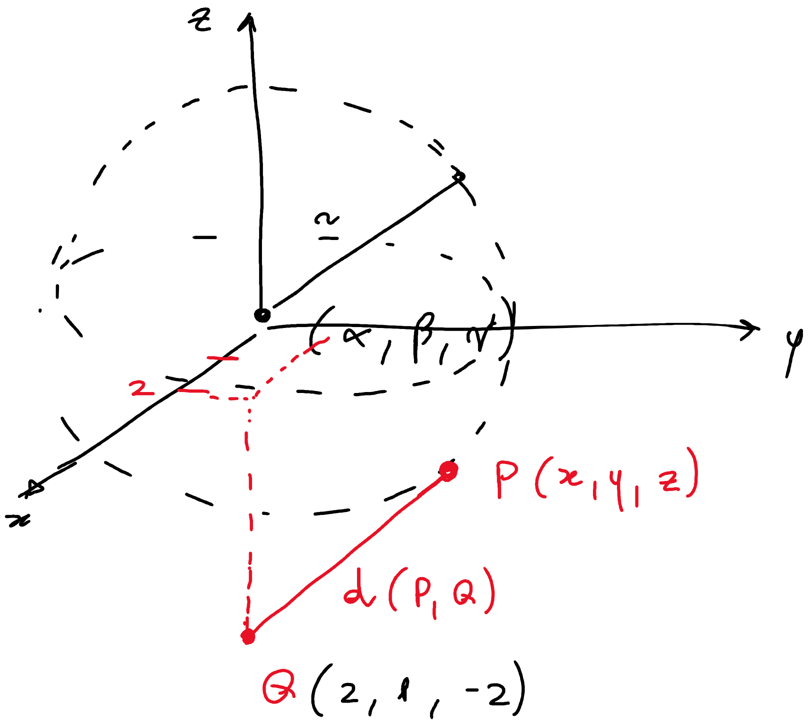
Find the minimum and maximum distance of the point $Q(2,1,-2)$ from the sphere

$$x^2 + y^2 + z^2 = 1.$$

$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$ circle



$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$$



$$d(P,Q) = \sqrt{(x-2)^2 + (y-1)^2 + (z+2)^2}$$

$$\max_{\min} \sqrt{(x-2)^2+(y-1)^2+(z+2)^2}$$

s.t. $x^2+y^2+z^2=1$

$$\mathcal{L} = \sqrt{\ ---} \quad - \lambda(\ ----\ )$$

$$\frac{1}{2\sqrt{\ ---}} \quad ---$$

Shortcut:

$\max \ d^2(P,Q)$ just to make disappear the root

s.t.: $x^2+y^2+z^2=1$

$$\max_{\min} (x-2)^2+(y-1)^2+(z+2)^2$$

s.t. $x^2+y^2+z^2=1$

CQ: $h(x,y,z) = x^2+y^2+z^2-1$

$\nabla h \neq \underline{0}$

$(2x, 2y, 2z) \neq (0,0,0)$

$(x,y,z) \neq (0,0,0)$

But the origin is not in the constraint set (i.e., the sphere) so the constraint qualification is automatically satisfied.