$$\begin{matrix}\max\\ \min\end{matrix} \quad (x-2)^2+(y-1)^2+(z+2)^2$$

$$\text{s.t.} \quad x^2+y^2+z^2=1$$

$$\text{CQ}: \quad h(x,y,z) = x^2+y^2+z^2-1$$

$$\nabla h \neq \underline{0}$$

$$(2x, 2y, 2z) \neq (0,0,0)$$

$$(x,y,z) \neq (0,0,0)$$

But the origin is not in the constraint set (i.e., the sphere) so the constraint qualification is automatically satisfied.

$$\mathcal{L}(x,y,z,\lambda) = f(x,y,z) - \lambda\big(h(x,y,z) - \underbrace{c}_{0}\big) =$$

$$= (x-2)^2+(y-1)^2+(z+2)^2 - \lambda\left(x^2+y^2+z^2-1\right)$$

$$\nabla \mathcal{L} = \underline{0}$$

$$\frac{\partial \mathcal{L}}{\partial x} = \cancel{2}(x-2) - \cancel{2}\lambda x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = \cancel{2}(y-1) - \cancel{2}\lambda y = 0$$

$$\begin{cases} x-2-\lambda x = 0 \\ y-1-\lambda y = 0 \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \cancel{2}(y-1) - \cancel{2}\lambda y = 0$$

$$\frac{\partial \mathcal{L}}{\partial z} = \cancel{2}(z+2) - \cancel{2}\lambda z = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\left(x^2+y^2+z^2-1\right) = 0$$

$\times(-1)$ $\times(-1)$

$$\begin{cases} y-1-\lambda y = 0 \\ z+2-\lambda z = 0 \\ x^2+y^2+z^2 = 1 \end{cases}$$

$$\begin{cases} x-2-\lambda x = 0 \\ y-1-\lambda y = 0 \\ z+2-\lambda z = 0 \\ x^2+y^2+z^2 = 1 \end{cases} \qquad \begin{cases} (1-\lambda)x = 2 \\ (1-\lambda)y = 1 \\ (1-\lambda)z = -2 \\ x^2+y^2+z^2 = 1 \end{cases}$$

$\lambda \neq 1$

$$x = \frac{2}{1-\lambda}$$

Particular case:

But what happens if $\lambda = 1$

The first equation becomes $(1-1)x = 2$; $0 = 2$

impossible.

If, vice versa, $\lambda \neq 1$, we have:

$$\begin{cases} x = \dfrac{2}{1-\lambda} \\[2ex] y = \dfrac{1}{1-\lambda} \\[2ex] z = -\dfrac{2}{1-\lambda} \\[2ex] \left(\dfrac{2}{1-\lambda}\right)^2 + \left(\dfrac{1}{1-\lambda}\right)^2 + \left(-\dfrac{2}{1-\lambda}\right)^2 = 1 \end{cases}$$

$$\frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 1$$

$$\frac{9}{(1-\lambda)^2} = 1$$

$$9 = (1-\lambda)^2$$

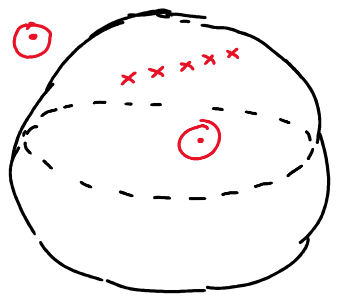
$$1 - \lambda = \pm 3$$

$$\lambda = 1 \mp 3 \quad \begin{cases} -2 \\ 4 \end{cases}$$

$$\begin{cases} x = \frac{2}{3} \\ y = \frac{1}{3} \\ z = -\frac{2}{3} \\ \lambda = -2 \end{cases} \qquad \begin{cases} x = -\frac{2}{3} \\ y = -\frac{1}{3} \\ z = \frac{2}{3} \\ \lambda = 4 \end{cases}$$

$$A\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \qquad B\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$$

$$x^2 + y^2 + z^2 = 1$$



closed

bounded

$\downarrow$

compact

$f(x, y, z) \in C$

Weierstrass' theorem

$$f(x,y,z) = (x-2)^2 + (y-1)^2 + (z+2)^2$$

$$f(A) = f\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) =$$

$$= \left(\frac{2}{3} - 2\right)^2 + \left(\frac{1}{3} - 1\right)^2 + \left(-\frac{2}{3} + 2\right)^2 =$$

$$= \left(-\frac{4}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2 =$$

$$= \frac{16}{9} + \frac{4}{9} + \frac{16}{9} = \frac{36}{9}$$

$$f(B) = f\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) =$$

$$= \left(-\frac{2}{3} - 2\right)^2 + \left(-\frac{1}{3} - 1\right)^2 + \left(\frac{2}{3} + 2\right)^2 =$$

$$= \left(-\frac{8}{3}\right)^2 + \left(-\frac{4}{3}\right)^2 + \left(\frac{8}{3}\right)^2 =$$

$$= \frac{64}{9} + \frac{16}{9} + \frac{64}{9} = \frac{128+16}{9} = \frac{138+6}{9} =$$

$$= \frac{144}{9}$$

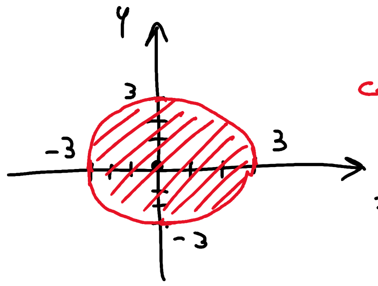
The distance between Q and A is minimum and the distance between Q and B is maximum

$$d(Q, A) = \sqrt{\frac{36}{9}} = \frac{6}{3} = 2$$

$$d(Q, B) = \sqrt{\frac{144}{9}} = \frac{12}{3} = 4$$

$$\max \quad x^4 + y^4 - 8(x^2+y^2)$$ continuous

$$\text{s.t.} \quad x^2 + y^2 \leq 9$$



compact

So we can apply Weierstrass' theorem

$$\mathcal{L}(x,y,\lambda) = x^4 + y^4 - 8(x^2+y^2) - \lambda(x^2+y^2-9)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0: \quad 4x^3 - 16x - 2\lambda x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0: \quad 4y^3 - 16y - 2\lambda y = 0$$

$$\lambda(x^2+y^2-9) = 0$$

$$x^2+y^2 \leq 9$$

$$\lambda \geq 0$$

case 1: $\lambda = 0$

$$\begin{cases} 4x^3 - 16x = 0 \\ 4y^3 - 16y = 0 \\ x^2 + y^2 \leq 9 \end{cases}$$

$4x^3 - 16x = 0 \qquad x^3 - 4x = 0$

$x(x^2 - 4) = 0 \qquad x(x-2)(x+2) = 0$

$x = 0, \quad x = 2, \quad x = -2$

$y = 0, \quad y = 2, \quad y = -2$

$A(0,0), \quad B(0,2), \quad C(0,-2),$
$D(2,0), \quad E(2,-2), \quad F(2,2), \quad G(-2,0)$
$H(-2,2), \quad I(-2,-2)$

$A: \quad 0^2 + 0^2 \leq 9 \qquad \checkmark$
$B: \quad 0 + 4 \leq 9 \qquad \checkmark$
$\vdots$
$J: \quad 4 + 4 \leq 9 \qquad \checkmark$

case 2: $\quad x^2 + y^2 = 9$

$$\begin{cases} 4x^3 - 16x - 2\lambda x = 0 \\ 4y^3 - 16y - 2\lambda y = 0 \\ x^2 + y^2 = 9 \\ \lambda \geq 0 \end{cases}$$

$$\begin{cases} 2x^3 - 8x - \lambda x = 0 \\ 2y^3 - 8y - \lambda y = 0 \\ x^2 + y^2 = 9 \\ \lambda \geq 0 \end{cases}$$

$$\begin{cases} \overbrace{x}\,\overbrace{(2x^2 - 8 - \lambda)} = 0 \\ y(2y^2 - 8 - \lambda) = 0 \\ x^2 + y^2 = 9 \\ \lambda \geq 0 \end{cases}$$

$x = 0$ or $2x^2 - 8 - \lambda = 0$

$y = 0$ or $2y^2 - 8 - \lambda = 0$

case 2.1 : $x = 0$ and $y = 0$

case 2.2 : $x = 0$ and $2y^2 - 8 - \lambda = 0$

case 2.3 : $2x^2 - 8 - \lambda = 0$ and $y = 0$

case 2.4 : $2x^2 - 8 - \lambda = 0$ and $2y^2 - 8 - \lambda = 0$

case 2.1

$$\begin{cases} x = 0 \\ y = 0 \\ 0^2 + 0^2 = 9 \\ \lambda \geq 0 \end{cases}$$ impossible

case 2.2

$$\begin{cases} x = 0 \\ 2y^2 - 8 - \lambda = 0 \\ 0^2 + y^2 = 9 \\ \lambda \geq 0 \end{cases} \quad \leftarrow \quad y^2 = 9 \qquad y \pm 3$$

$$\begin{cases} x = 0 \\ 18 - 8 - \lambda = 0 \\ y = \pm 3 \\ \lambda \geq 0 \end{cases} \qquad \begin{cases} x = 0 \\ \lambda = 10 \\ y = \pm 3 \\ \lambda \geq 0 \end{cases}$$

$J(0, 3) \qquad K(0, -3)$

case 2.3

In a similar way, we get

$L(3, 0)$ and $M(-3, 0)$

case 2.4

$$\begin{cases} 2x^2 - 8 - \lambda = 0 \\ 2y^2 - 8 - \lambda = 0 \\ x^2 + y^2 = 9 \\ \lambda \geq 0 \end{cases} \qquad \begin{cases} \lambda = 2x^2 - 8 \\ \lambda = 2y^2 - 8 \\ x^2 + y^2 = 9 \\ \lambda \geq 0 \end{cases}$$

$$\begin{cases} \lambda = 2x^2 - 8 \\ 2x^2 - \cancel{8} = 2y^2 - \cancel{8} \\ x^2 + y^2 = 9 \\ \lambda \geq 0 \end{cases} \qquad x^2 = y^2 \qquad x = \pm y$$

$$\begin{cases} \lambda = 2x^2 - 8 \\ y = \pm x \\ 2x^2 = 9 \\ \lambda \geq 0 \end{cases} \qquad \begin{cases} \lambda = 2 \cdot \frac{9}{2} - 8 = 1 \\ y = \pm \frac{3}{\sqrt{2}} \\ x = \pm \frac{3}{\sqrt{2}} \\ \lambda \geq 0 \end{cases} \qquad ✓$$

$N \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}} \right)$

$O \left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}} \right)$

$P \left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}} \right)$

$Q \left(-\frac{3}{}, -3 \right)$

$$Q\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$$

At this point, you substitute the coordinates of ALL the candidate points into the objective function and select the one that maximizes it.

For example:

$$f(x, y) = x^4 + y^4 - 8(x^2 + y^2)$$

$$f(I) = f(-2, -2) =$$

$$= (-2)^4 + (-2)^4 - 8\left((-2)^2 + (-2)^2\right) =$$

$$= 16 + 16 - 8(4 + 4) =$$

$$= 32 - 8 \cdot 8 = 32 - 64 = -32$$

etc...