International Finance and Economics
Dept. of Economics and Law

Mathematical methods for economics and finance Mod A

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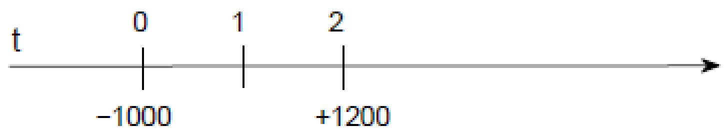
PART A1
Basic instruments in mathematical finance

INITIAL PROBLEM: Consider an amount of money w(t0) that is available at time t0 (for instance today). Consider that w(t0) wants to be invested at time t0 until time t1 (i.e. the length of the investment period is t=t1-t0). Then, at time t1, the new disposable amount is w(t1) and it must be w(t1)>w(t0).

To act this strategy it is necessary to find somebody who needs to have w(t0) at time t0 while accepting to pay w(t1) at time t1.

REPRESENTATION

EX1: INVEST today 1000 euro to receive in 2 years 1200 euro; **graphic representation**:



On the other side there may exist a person who wants to **BORROW**: receive today 1000 euro and give back in 2 years 1200 euro; **flow-deadline** representation: OF={(1000,-1200);(0,2)}

THIS IS A FINANCIAL OPERATION!

TWO POINTS OF VIEW:

Borrowing (all entries and then all exits) or **Investments** (vice versa)

DEF. SIMPLE FINANCIAL OPERATION

This is a financial operation composed only by ONE entry and ONE exit, that is one amount to be paid (-) and one amount to be received (+) (as in the previous example EX1) so that the **flow vector has only two elements**.

DEF. COMPLEX FINANCIAL OPERATION

This is a financial operation composed by more than two amounts, so that the **flow vector has more than two elements**.

EX2: BORROW today 1000 euro, and pay 500 euro in one year and 600 euro in 2 years; flow-deadline representation:

OF={(1000,-500,-600);(0,1,2)} is a complex financial operation

Notice that: when it is not differently specified then the time is expressed in terms of years.

MAIN GOAL: To understand the way in which money changes its value over time.

We first focus on **simple financial operations**.

Two fundamental questions:

1) What is the **future value** of an amount invested or borrowed today?
2) What is the **present value** of an amount to be paid or received at a certain time in the future?

To give an answer we need to know the **financial rules** to be considered: that is which is the function to be used to determine the value of money at a given time if it is known its value at a different time.

Several **financial laws** can be used, but we recall the most employed ones:

1. RULE OF SIMPLE INTEREST
2. RULE OF COMPOUNDING-INTEREST

RULE OF SIMPLE INTEREST

Def.

Consider an amount of money at time t=0, w(0), called **PRINCIPAL**, and consider a rule such that the FUTURE VALUE is given by the principal plus **an interest that is attracted only by the principal**. Let t=0,1,…,n, then:

$$w(1) = w(0) + iw(0) = w(0)(1 + i)$$

$$w(2) = w(1) + iw(0) = w(0) + iw(0) + iw(0) = w(0)(1 + 2i)\ldots$$

$$\ldots w(n) = w(0)(1 + in), n \in \mathbb{N}$$

Where **i>0 is the interest rate related to one period** while **n is the number of periods**.

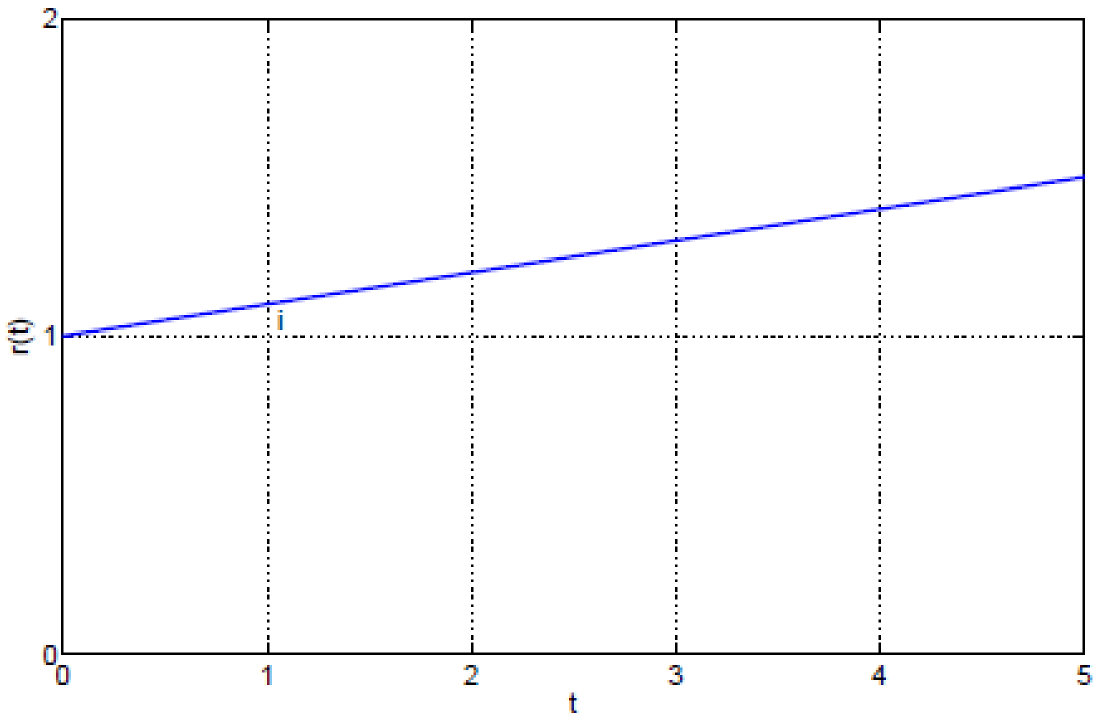
If t is a real number denoting the length of the operation, then we obtain the following formula **(*)**

$$\boxed{w(t) = w(0)(1 + it)}$$

If we consider 1 euro at the initial time, w(0)=1, then we can define

$$r(t) = \frac{w(t)}{w(0)} = (1 + it)$$

that is the **GROWTH FACTOR**. Its graph is linear.



EX3: Consider the rule of simple interest. If today we invest 2000 euro at an interest rate (annual) of 10%=0.1, then in 3 years we will receive

$$M=2000(1+0.1*3)=2600 \text{ euro.}$$

Hence, according to this law, 2600 euro in 3 years is financially equivalent to 2000 euro today.

FIRST QUESTION: the time unit

To correclty apply the formula, it is very important to be **coherent** when considering the number of periods (**n**) and the interest rate (**i**) to be used. In fact they must refer to the same time unit!

EX4: The life of the investment is 18 **months** and it is given the **annual** interest rate i=10%. To correcly find the future value we can procede as follows:

1.Since it is given an **ANNUAL interest rate**, we can change the number of months (18) into the **number of years** (18/12=1.5).

2.Since it is given the length in terms of **number of months**, we can change the annual interest rate into the **monthly interest rate**. .....

DEF. Equivalent interest rates

Let i_m be the interest rate related to 1-mth of year and let i_n be the interest rate referred to 1-nth of year. Then i_m and i_n **are said to be equivalent** iff, when applied to the same initial value for the same period they produce the same final value.

Equivalent interest rates under the simple interest rule: According to the previous definition, consider the principal of 1 euro and a one-year period, then, taking into account the rule of simple interest, the following formula holds:

$$m i_m = n i_n \Rightarrow i_m = \frac{n}{m} i_n.$$

Notice that $i_1=i$ is the annual interest rate, hence:

$$\Rightarrow i_m = \frac{i}{m}.$$

EX4: …from the previous example, we have that the annual interest rate of 0.1 is equivalent to $i_{12}=0.1/12=0.00833$.

EX5: We have to invest 5000 euro for 15 months and the interest rate referred to a period of 6-months (semi-annual) is $i_2=0.05$.
The final value is given by

$$M_1 = 5000(1 + 0.05 \cdot \frac{5}{2}) = 5625$$

where the length of 15 months is replaced by the number of semesters 15/6=5/3,

or **alternatively**, by

$$M_2 = 5000(1 + 0.0083 \cdot 15) = 5625$$

where the semi-annual interest rate has been converted into the monthly interest rate

$$2i_2 = 12i_{12} \quad \Rightarrow \quad i_{12} = \tfrac{2}{12} i_2 = 0.0083$$

SECOND QUESTION: the present value

If it is known the value at a future time t, then it is possible to determine its present value (at time t=0) simply solving the previous formula

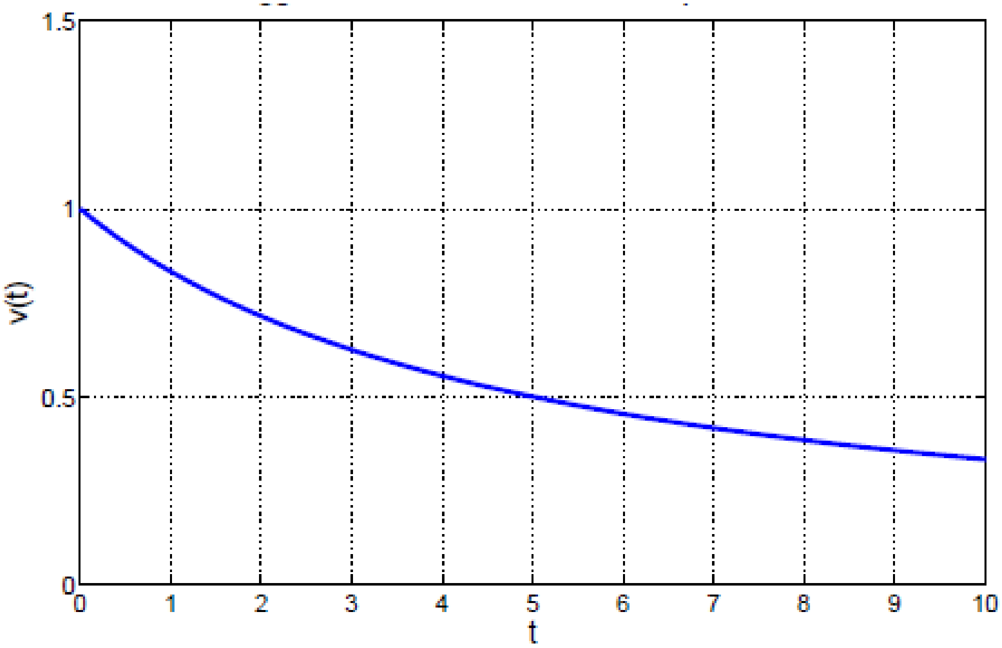
$$w(t) = w(0)(1 + it)$$

for the principal, thus obtaining

$$w(0) = \frac{w(t)}{(1 + it)}$$

where the **discount factor** is:

$$v(t) = \frac{1}{r(t)} = \frac{1}{1 + it}$$



EX6: Bob holds a promissory note of 3000 euro at 4 years, but he wants to discount it to obtain today its present value for liquidity purpose. The bank applies the simple interest rule with a monthly interest rate of 1%.

Since it is given i_{12}=0.01 then the annual equivalent interest rate is i=12*0.01=0.12 and the length of the operation is 4 years.

Hence the present value is w(0)=3000/(1+0.12*4)=2027.

HOMEWORKS

1. Consider the simple interest rule and answer. (a) The bimonthly interest rate of 5% is equivalent to the annual interest rate ____________. (b) If $i_{1/2}$=0.2 then i_2=________. (c) If the interest rate is 6% then i_3=____________.

2. We have 500 euro today and we want to invest this amount for 6 years with simple interest rule. For the first 3 years it is applied the interest rate of 8% while for the last 3 years it is used i_4=0.01. Determine the final value of the investment.

3. For liquidity purpose we want to obtain today the present value of 10000 euro that will be disposable in 90 days. Determine its present value with simple interest rule by considering that i=0.12.

[Usually we consider 1 month=30 days and 1 year=360 days]

RULE OF COMPOUNDING INTEREST (OR EXPONENTIAL INTEREST)

Normally the interest already earned can be reinvested to attract even more interest.

Def.

Consider an amount of money at time t=0, w(0), and assume that **the interest earned will be added to the principal periodically** (hence interests will be attracted not just by the original amount but also by all the interest earned so far) as follows:

$$w(1) = w(0) + iw(0) = w(0)(1+i)$$

$$w(2) = w(1) + iw(1) = w(1)(1+i) = w(0)(1+i)^2 \ldots$$

$$\ldots w(n) = w(0)(1+i)^n, n \in \mathbb{N}$$

This rule is called **compounding interest method** or **exponential rule** and the final formula is **(**)**

$$\boxed{w(t) = w(0)(1+i)^t}$$

where t is a real number representing the length of the operation while i is the interest rate.

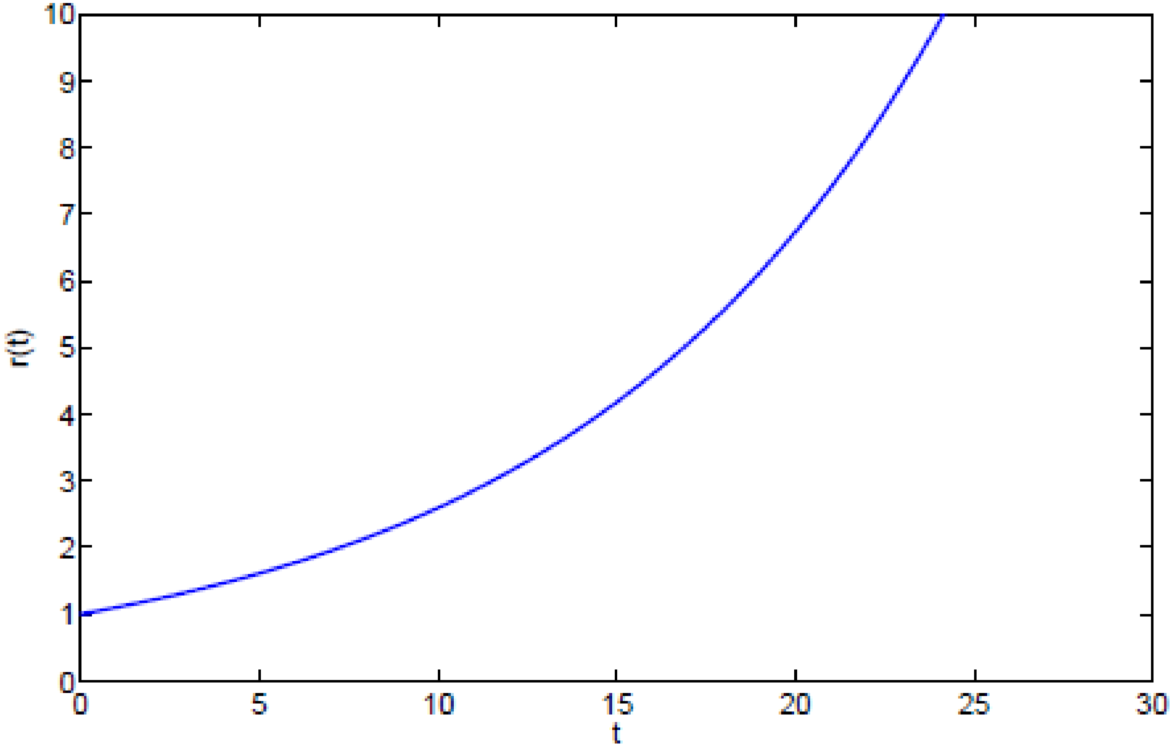
Again variables t and i must refer to the same time-unit.

EXPONENTIAL RULE

If we consider 1 euro at the initial time, w(0)=1, then we can define

$$r(t) = (1+i)^t$$

that is the **GROWTH FACTOR**. Its graph is exponential.



EX7: Let i=10%. Then with the exponential rule 1000 euro today is equivalent, after 8 months, to

$$M = 1000(1+0.1)^{\frac{8}{12}} = 1000(1.1)^{\frac{2}{3}} = 1065.6$$

Notice that, since we are using an interest rate referred to one year, then also the length of the investement must be expressed in terms of years, that is 8 months=8/12 years.

FIRST QUESTION: the time unit

Given an interest rate referred to a given time-unit (for instance one-month, i_{12}), how to determine the equivalent interest rate referred to a different time-unit (for instance one year, i) by considering the exponential rule?

DEF: Given the definition of equivalent interest rates, then i_m and i_n **are equivalent according to the exponential rule** iff

$$(1+i_m)^m = (1+i_n)^n \Rightarrow i_m = (1+i_n)^{n/m} - 1$$

EX8: Consider the exponential rule and let i_2=0.06. To determine the annual equivalent interest rate we place:

$$(1+i)=(1+i_2)^2 \Rightarrow i=(1.06)^2-1=0.1236$$

while to obtain the monthly equivalent interest rate we place:

$$(1+i_{12})^{12}=(1+i_2)^2 \Rightarrow i_{12}=(1+i_2)^{2/12}-1 \Rightarrow i_{12}=(1.06)^{1/6}-1=0.00976$$

Notice that when the time and the interest rate are referred to different time-units, then they must be expressed into the same unit. To the scope it is possible to change the time-unit of t or, equivalently, the time-unit of the interest rate i.

EX9: Let w(0)=2000, i_2=0.05, t=20 months and consider the exponential rule. To calculate the future values the following procedures are equivalent.

1. Consider that 20 months=20/6 semesters and use i_2.

$$t=\frac{20}{6}=3.\bar{3} \Rightarrow M_1=2000(1.05)^{3.\bar{3}}=2353.21.$$

2. Change i_2 into the equivalent monthly interest rate and consider t=20 months.

$$i_{12}=(1.05)^{\frac{1}{6}}-1=0.00816 \Rightarrow M_2=2000(1.00816)^{20}=2352.98.$$

The little difference is due to the approximantion.

SECOND QUESTION: the present value

If it is known the value at a future time t, then it is possible to determine its present value (at time t=0) simply solving the previous formula

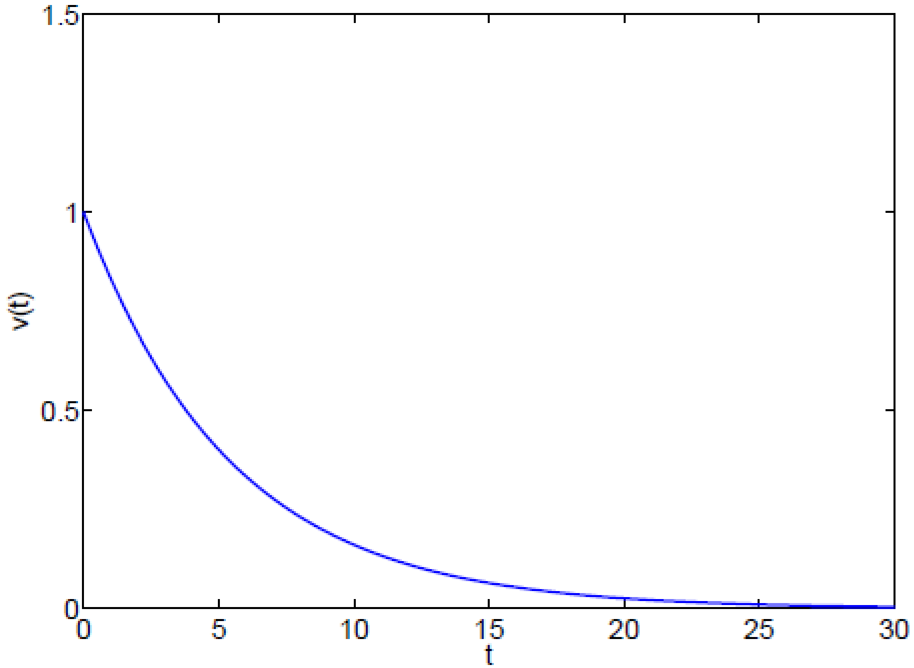
$$w(t) = w(0)(1+i)^t$$

for the principal, thus obtaining

$$w(0) = w(t)(1+i)^{-t}$$

where the **discount factor** is:

$$v(t) = (1+i)^{-t}$$

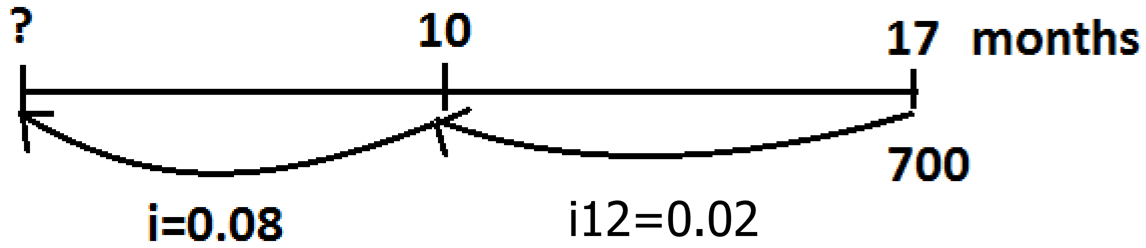


EX10: We will receive 10000 euro in one year and it is given a semiannual interest rate of 0.05 in the exponential interest rule. The present value is:

$$C = 10000(1.05)^{-2} = 9070.3.$$

EX11: The amount of 700 euro will be disposable in 17 months. We wants to determine its present value with the exponential rule by considering that: (1) during the first 10 months it is given i=0.08.(2) during the last period it is applied the monthly interest rate 2%

Graphically:



From i=0.08 it can be obtained

$$i_{12} = (1.08)^{\frac{1}{12}} - 1 = 0.0064$$

Hence

$$w(0) = 700(1.02)^{-7}(1.0064)^{-10} = 571.73$$

HOMEWORKS

4. Consider the compound interest rule and answer to the following questions. (a) If the quarterly interest rate in 4% then the equivalent annual interest rate is ___________. (b) Let i_2=0.01 then i=_____________, i_3=_____________ and i_6=______________.

5. Consider 1000 euro disposable at time t=2 and a montly interest rate of 0.5% with the exponential interest rule. Determine the equivalent amount of money at times t=1.5 and t=6.

6. The amount of 8000 euro will be disposable in 27 months. Consider the exponential interest rule with interest rate 8% and calculate the present value. The present value is then invested at the same interest rate for 18 mnths with the simple interest rule. Determine the final value.

7. Let be given 5000 euro today to be invested for 15 months, with the simple interest rule and i=9% OR with the exponential interest rule and i2=4%. Which is the most convenient option?

[When investing it is preferred the highest final value, when borrowing it is preferred the highest present value]

MatLab= MATrix LABoratory (environment for the scientific calculation and numerical simulations)

MatLab Desktop :

- **Command Window:** (to insert commands and instructions, >> is the prompt)
- **Command History:** chronological sequence of the executed instructions
- **Workspace:** operative memory containing the variables, array
- **Current Directory:** containing all the files and folders

Save a real number

To assign a name to a number you must use the symbol **=**

Define variable a with value 25

```
>> a=25
```

Such variable will be saved in the Workspace!

Notice: MatLab is *case sensitive*

Notice: ; after the command: the result of the instruction will not appear in the command window

Try to:

Save in MatLab the following real numbers:

$$A = \frac{12}{5}; B = 3^6; C = 2.1 \cdot 7$$

Notice: the point separates decimals

And calculate:

$$a = A + \frac{B}{C}; b = \frac{C - a}{B}; c = bB^A$$

Some initial commands

>>clc cleans command window

>>clear cleans workspace

>> ↑ recall the last executed command

>>doc "argument" consults MatLab help on the «argument»

>>format long activates 14 decimals format

>>format short activates 4 default decimal format

Use elementary functions

To save number e^2 ...we need to know the syntax of the required function!

Notice: >>doc elfun to consult MatLab help on elementary functions. Thus obtaining

```
>> w=exp(2)
```

Try to

Save in MatLab the following real numbers:

$$x = \ln(4); y = \frac{1}{8}; z = -6$$

And calculate:

$$p = e^{x} - \ln 2; q = \sqrt{y}z^{3}; r = |-10 + 2z|$$

EX12: You can use MatLab to calculate the value of a payment: consider 2200 euro disposable in 15 months and the rule of exponential interest with i=11%. Determine its present value

First we choose to determine the monthly interest rate: $i_{12} = (1.11)^{1/12} - 1$

With MatLab

```
>> i12=(1.11)^(1/12)-1

i12 =

   0.0087
```

Then compute the present value: $PV = 2200(1+i_{12})^{-15}$

With MatLab

```
>> pv=2200*(1+i12)^-15

pv =

  1.9309e+03
```

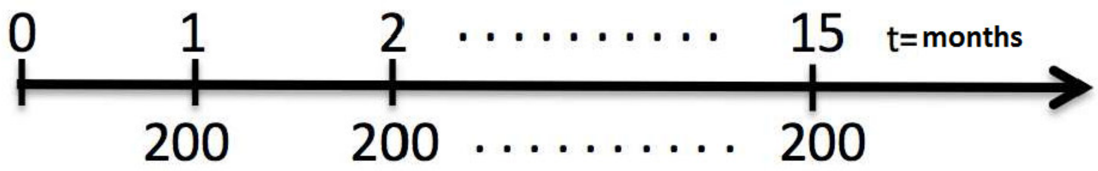
HENCE the present value is: 1930.9

ANNUITY

Def. Annuity: An annuity is a sequence of **finitely many payments** of a fixed amount (R) **due at equal time intervals** for n periods. This is a complex financial operation.

Def.: An annuity is called **ORDINARY-ANNUITY** or **annuity-immediate** if the payments are made at the end of the payment periods, while it is called **ANNUITY-DUE** if the payments are made at the beginnning of the payment periods

EX13: Annuity that starts today of amount 200 paid montly for 15 months: this is an annuity-immediate



MAIN GOAL: we wants to know the value of the annuity at time 0, called **present value**, or the accumulated value of the annuity at time n, called the **future value**, by considering the compounding interest rule where **i is the rate of interest per period** (for instance for a quarterly annuity we need to use i_4, for a semi-annual annuity the rate i_2 must be used…)

ORDINARY ANNUITY

Consider an ordinary annuity, then its **present value** is given by the sum of the present values of each payment.

Denote with w(R,0) the present value of an ordinary annuity of amount R. If R=1 we have a unitary ordinary annuity and we define

$$v = (1+i)^{-1}$$

then

W(1,0)=
$$\begin{aligned} a_{\overline{n}|i} &= v + v^2 + v^3 + \cdots + v^n \\ &= v \times \left[\frac{1-v^n}{1-v}\right] \\ &= \frac{1-v^n}{i} \\ &= \frac{1-(1+i)^{-n}}{i}. \end{aligned}$$

Called «a angle n at i»

And consequently, $w(\underline{R}, 0) = R \cdot a_{\overline{n}|i}$ is the present value of an ordinary annuity of amount R

From the present value of an annuity is is immediate to obtain the **future value** (or the value at any given time):

$$w(\underline{R}, n) = w(\underline{R}, 0)(1+i)^n = R\, a_{\overline{n}|i}(1+i)^n =$$

$$R\frac{(1+i)^n - 1}{i} = R\, s_{\overline{n}|i}$$

EX14: Calculate the present value of an ordinary annuity of amount \$100 paid annually for 5 years at the rate of interest of 9% . Also calculate its future value at time 5.

Present value: $100\, a_{\overline{5}|0.09} = 100 \times \left[\frac{1-(1.09)^{-5}}{0.09}\right] = \$388.97$

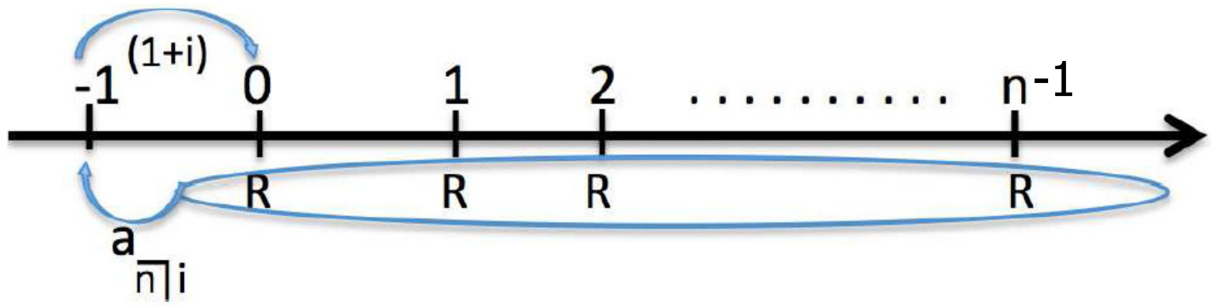
Future value: $(1.09)^5 \times (100\, a_{\overline{5}|0.09}) = (1.09)^5 \times 388.97 = \$598.47.$

or alternatively: $100\, s_{\overline{5}|0.09} = 100 \times \left[\frac{(1.09)^5 - 1}{0.09}\right] = \$598.47.$

ANNUITY DUE

Notice that: if we have an annuity-due then **the first payment is made at time t=0** while the last payment is made at time t=n-1.

Graphically:



From the formula of the present value of an oridinary annuity, we can calculate the **present value of an annuity-due**:

$$w(\underline{R}, 0) = Ra_{\overline{n}|i}(1+i) = R\ddot{a}_{\overline{n}|i}$$ Called «a dots angle n at i»

And from the present value, the final value can be easily obtained.

EX15: Find the present value of an annuity-due of 200 euro per quarter (every three-months) for 2 years, if the monthly interest rate is 0.08/12.

First of all the monthly interest rate must be converted into the quarterly interest rate i_4 according to the compound interest method:

$$\left[1+\frac{0.08}{12}\right]^3-1=2.01\%.$$

Since we have a quarterly annuity for 2 years the number of payments is 4x2=8

Then the present value of the annuity is:

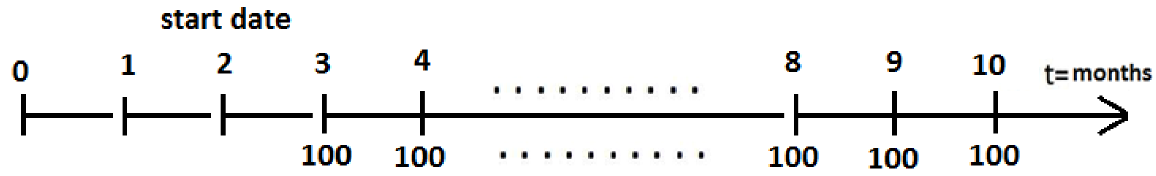
$$200\,\ddot{a}_{\overline{8}|0.0201}=200\frac{1-(1.0201)^{-8}}{0.0201}(1.0201)=1493.9$$

Def. Deferred Annuity: A deferred annuity is one for which the first payment starts some time in the future.

The present value (or future value) of a deferred annuity can be simply obtaining from that of a not deferred annuity by compounding for an oportune number of periods.

EX16: Consider a monthly annuity-immediate of 100 euro starting at 2 months and ending at 10 months. Determine its present value given i=0.1.

Graphic representation:

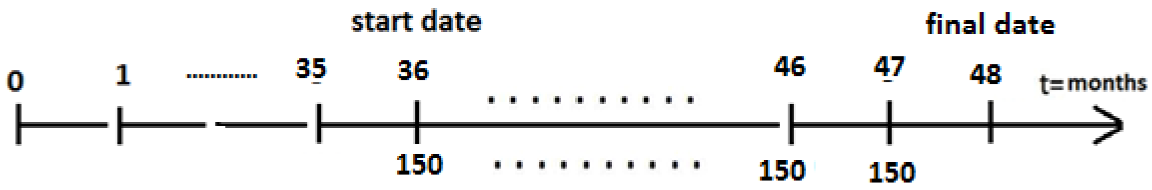


Data: number of payments **n=8**, each payment 100, monthly interest rate=**0.008**. If we compute $Ra_{\overline{n}|i}$ we obtain the value of the annuity at time t=2 hence, to obtain the present value we have to compute:

$$100\frac{1-(1.008)^{-8}}{0.008}(1.008)^{-2}=759.75$$

EX17: Consider a monthly annuity-due of 150 euro starting at 3 years, with 12 payments, and interest rate 8%. Determine its present value and future value (when the annuity ends).

Graphic representation:



Data: number of payments **n=12**, each payment 150, monthly interest rate=**0.0064**. If we compute $Ra_{\overline{n}|i}$ we obtain the value of the annuity at time t=35 (months) hence,

to obtain the **present value** (at time t=0) we have to discount for 35 periods:

$$150\frac{1-(1.0064)^{-12}}{0.0064}(1.0064)^{-35}=1381.6$$

to obtain the **final value** (at time t=48) we have to capitalize for 13 periods:

$$150\frac{1-(1.0064)^{-12}}{0.0064}(1.0064)^{13}=1876.7$$

EX18: You can use MatLab to calculate the value of an annuity: consider an annuity-due of 625 euro, payable every year , ending in 10 years, monthly interest rate 0.5%. Find the present value.

First determine the annual interest rate: $i = (1.005)^{12} - 1$

With MatLab

```
>> i=(1.005)^12-1
i =
   0.0617
```

Then compute the present value: $PV = 625 \frac{1-(1+i)^{-10}}{i}(1+i)$

With MatLab

```
>> PV=625*(1-(1+i)^-10)/i*(1+i)
PV =
  4.8452e+03
```

HENCE the present value is: 4845.2

HOMEWORKS

8. Calculate the present value of an ordinary annuity of amount 100 euro payable quarterly for 10 years at the annual rate of interest of 9%. Also calculate its future value at the end of 10 years.

9. Find the present value of an annuity-due of 200 euro per semester for 2 years, if the semi-annual interest rate is 5%.

10. Consider an immediate-annuity of 300 euro, payble every 6 month for 30 months, interest rate 7%. Determine its presente value and final value.

11. Find the value after 5 years of an annuity-due of 400 euro payable every 2 months for 17 years given i_2=0.5.

12. Consider a yearly annuity-immediate of 500 euro starting at 5 year with 11 payments. Find the present value and the final value given $i_{1/2}$=18%.

13. Consider an annuity-due of 160 euro payable every two-monts, starting at 2 years and ending at 4 years. Find the present value and the final value (at the end of the annuity, i.e. after 4 years) given i_{12}=2%.

[You can use MatLab for calculations]

Def. Perpetuity: A perpetuity is an annuity with no termination date,

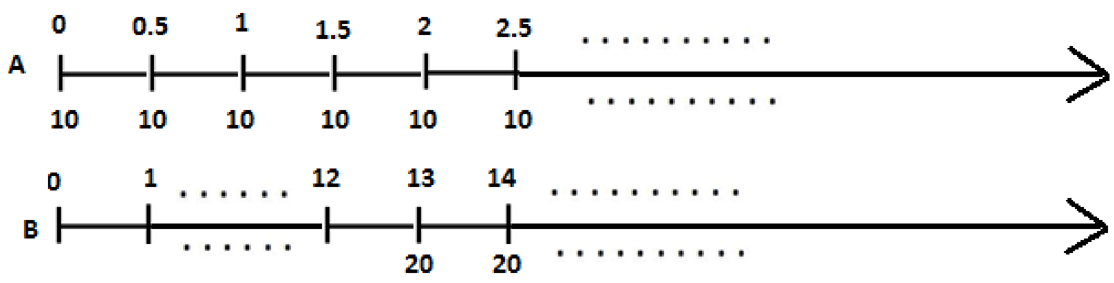
i.e., $n \to \infty$.

Similar definitions to those related to an annuity can be given.

Def.: A perpetuity is called **ORDINARY-PERPETUITY** or **perpetuity-immediate** if the payments are made at the end of the payment periods, while it is called **PERPETUITY-DUE** if the payments are made at the beginnning of the payment periods

Def. Deferred perpetuity: A deferred perpetuity is one for which the first payment starts some time in the future.

EX19:



A is a semi-annual perpetuity-due, **B** is an annual perpetuity-immediate deferred 12 years (or perpetuity-due deferred 13 years)

MAIN GOAL: Also for a perpetuity it is of interest to determine its **present value**, while obviously the final value cannot be calculated.

PRESENT VALUE OF A PERPETUITY

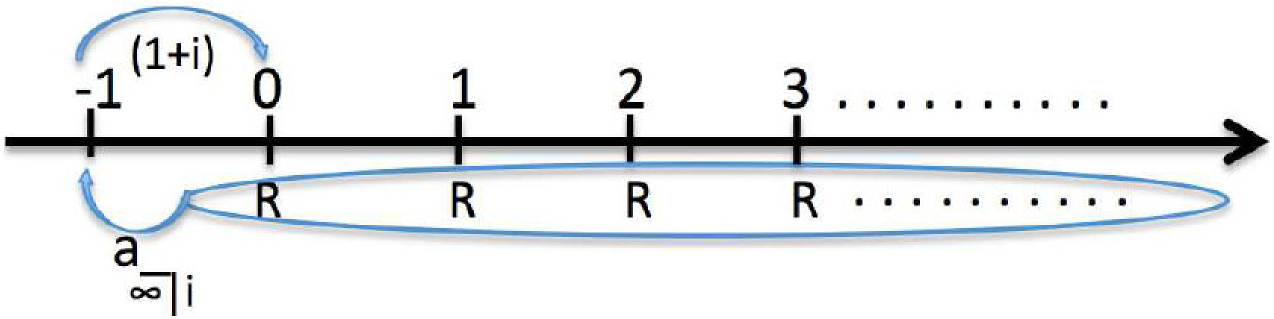
To calculate the present value of a perpetuity, we note that, as $v < 1, v^n \to 0$ as $n \to \infty$. Thus, from the formula of the present value of an annuity

we obtain

$$\lim_{n \to +\infty} \frac{1-(1+i)^{-n}}{i} = \frac{1}{i} = a_{\overline{\infty}|i}\,.$$

Called «a angle infinity at i»

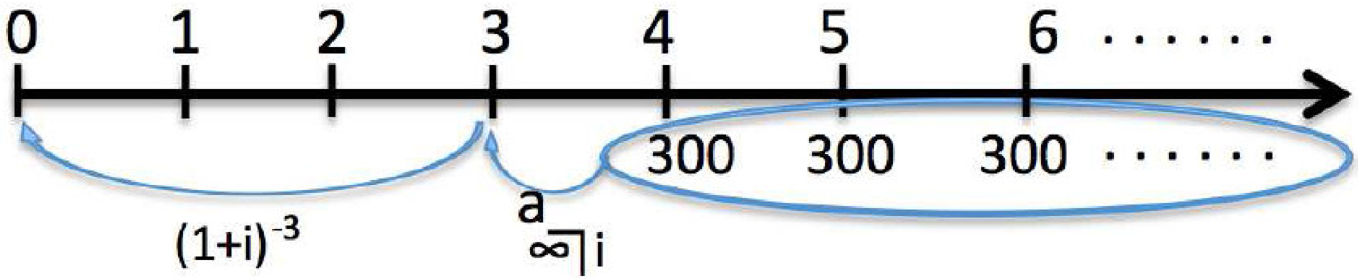
while for the case when the first payment is made immediately, we have



$$R a_{\overline{\infty}|i}(1+i) = R \ddot{a}_{\overline{\infty}|i}$$

and similarly the formula for the deferred case can be obtained.

EX20: Consider an annual perpetuity of 300 euro, the first payment is made at the end of the third year and the interest rate is 0.06. The present value is obtained as follows.



$$300 \cdot \frac{1}{i} \cdot (1+i)^{-3} = 300 \cdot \frac{1}{0.06} \cdot (1.06)^{-3} = 4198.09.$$

EX21: Consider a monthly perpetuity-due of 200 euro, the semi-annual interest rate is 0.07. Determine the present value and the value after 5 months.

We need to use the monthly interest rate:

$$(1+i_2)^2 = (1+i_{12})^{12} \Rightarrow i_{12} = (1.07)^{1/6} - 1 = 0.0113$$

The present value is obtained as follows: $PV = 200 \frac{1}{0.0113}(1.0113) = 17899$

so the value at time t=5 is simply obtained:

$$PV5 = PV(1+i_{12})^5 = 17899(1.0113)^5 = 18934$$

HOMEWORKS

14. Calculate the present value of an ordinary perpetuity of amount 100 euro payable quarterly at the annual rate of interest of 9%.

15. Find the present value of a perpetuity-due of 200 euro per semester starting after 3 semesters, if the monthly interest rate is 1%.

16. Consider an immediate-perpetuity of 300 euro, payble every 6 month starting after 4 years, interest rate 7%. Determine its value after 10 years.

17. Consider a semi-annual perpetuity due of amount 50 being i2=0.05. Determine the PV.

18. Consider a perpetuity o amount 10, payment every year, first payment at time 3. Determine the PV and the value at time 10 being i2=0.04.

[You can use MatLab for calculations]

INTERNAL RATE OF RETURN

Consider a project represented in terms of flows-deadlines as follows:

$$OF = \{(x_0, x_1, x_2, \ldots, x_n); (t_0, t_1, t_2, \ldots, t_n)\}$$

Where $x_j>0$ if at time t_j there will be an entry while $x_j<0$ if we have an output at time t_j.

We want to establish which is the interest rate associated to OF so we define the following.

Def: Internal rate of return (IRR) or yeald rate. The IRR is the rate of interest (if it exists unique) such that the present value of the financial project is equated to zero.

Notice that: 1. at the IRR the present values of the entries is equal to the present value associated to the exits; 2. normally it is required to calculate the IRR referred to one year.

Given the following OF

$$OF = \{(x_0, x_1, x_2, ..., x_n); (0, 1, 2, ..., n)\}$$

Then the IRR exists iff the following polynomial admits a unique zero

$$P_n(v) = x_0 + x_1 v + x_2 v^2 + ... + x_n v^n, \quad v = \frac{1}{1+i} \in (0, 1)$$

EX22: Determine the IRR of the following financial project:

$$\text{L'OF} = \{(-100, -100, +220); (0, 1, 2)\}$$

The present value is given by: $PV = -100 - 100(1+i)^{-1} + 220(1+i)^{-2}$

And the present value is equal to zero iff:

$$-100 - 100v + 220v^2 = 0, \quad v = \frac{1}{1+i}$$

$$\Rightarrow \quad 2.2v^2 - v - 1 = 0$$

$$\Rightarrow \quad v_{1,2} = \frac{1 \pm \sqrt{1+8.8}}{4.4} = \frac{1 \pm 3.13}{4.4}$$

$$v_1 = 0.9386 \in (0,1) \quad \text{e} \quad v_2 = -0.4841 \notin (0,1)$$

From v*=0.9386 we obtain i*=1/v*-1=0.0654. This is the IRR.

EX23: Determine the IRR of the following financial projects:

$$OF_A = \{(100, -120); (0, 5)\}$$

$$OF_B = \{(100, -60, -50); (0, 0.5, 1)\}$$

A) $$100 - 120v^5 = 0 \quad \Rightarrow \quad v^5 = \frac{100}{120} = 0.8333$$

$$\Rightarrow \quad (1+i)^{-5} = 0.8333 \quad \Rightarrow \quad i^* = (\frac{1}{0.8333})^{\frac{1}{5}} - 1 = 0.0371$$

B) $$100 - 60(1+i)^{-0.5} - 50(1+i)^{-1} = 0$$

$$v = (1+i)^{-0.5}$$

$$-5v^2 - 6v + 10 = 0$$

$$v = 0.9362$$

$$i^* = (0.9362)^{-2} - 1 = 0.1409.$$

NOTICE THAT: There is generally no analytic solution for i to the equation PV=0 when n > 2, and numerical methods have to be used. MatLab can be used to compute the answer easily.

You can use MatLab to calculate IRR of a cash flow

VECTORS

A **vector** composed by n elements is given by n ordered real numbers

Row vector Ex: $\underline{x}=(1, 3, -4, 11)$ composed by 4 elements (dimension 4)

Column vector Ex:

$$\underline{y}= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$ composed by 3 elements (dimension 3)

A real number z=(133) is a row or column vector having one element

Save a vector

ROW VECTOR: list the elements separated by a **space** or a **comma ,** inside square brackets

Save the row vector (10,-1,5)

```
>> A=[10 -1 5]
```

COLUMN VECTOR: list the elements separated by a **semicolon ;** inside square brackets (or write a row vector and put **apostrophe '** at the end)

Save the column vector with elements 3, 7, 9

```
>> B=[3;7;9]
```

You can use MatLab to calculate IRR of a cash flow: the command **irr** calculates the internal rate of return for a series of **periodic** cash flows. The obtained IRR is related to the period of the payments.

Syntax: Return = irr(CashFlow)

Where **CashFlow** is a **row vector** containing a stream of periodic cash flows. The first entry in CashFlow is the initial amount (at time t), while the following entries are the payment at time t+1,t+2,…,t+n.

EX 24: You can use MatLab to calculate the IRR: Find the internal rate of return. The initial investment is 100000 and the following cash flows represent the yearly income from the investment. Year 1:10000, Year 2: 20000, Year 3:30000, Year 4: 40000, Year 5 : 50000.

Since this flow is periodic we can use the command irr and since the periodicity is yearly then we will obtain the annual IRR.

First of all we define the cash-flow as a vector:

```
>> CF=[-100000 10000 20000 30000 40000 50000]

CF =

   -100000      10000      20000      30000      40000      50000
```

Then we give the instruction to obtain i (the annual IRR):

```
>> i=irr(CF)

i =

   0.1201
```

Consider the following cash flow {(-30,10,25);(1,2,2.5)} that **is not periodic**, then we need to use a different command and to specify to MatLab the dates.

You can use MatLab to calculate IRR of a non periodic cash flow: the command **xirr** calculates the internal rate of return for a series of non-**periodic** cash flows. The given IRR is annual and in default it is computed on actual basis (actual number of days)

Syntax: Return = xirr(CashFlow, CashFlowDates)

The additional input **CashFlowDates** ia a **column vector** of serial date numbers of the same size as CashFlow.

Each element of CashFlowDate represents the dates of the corresponding column of CashFlow. It must be defined as a string **'mm/gg/aaaa'** and MatLab will convert it in a number.

Furthermore, recall that to write a column vector it must be used square brakets with elements separated by semicolon.

For the initial case: dates (1,2,2.5) can be defined as:

CFD=['01/01/2000';'01/01/2001';'07/01/2001']

Where we fixed a starting date and we opportunely determined the following dates

Notice that: number of days in both a period and a year is the actual number of days.

EX 25: You can use MatLab to calculate the IRR for a non periodic cash flow:
Find the internal rate of return. The project is as follows where the time is in terms of months. OF={(-30,-20,28,32);(0,2,2.5,3)}

We first define the row vector of the cash flow:

```
>> cf=[-30 -20 28 32]
cf =
  -30   -20    28    32
```

Then we define the column vector of the dates, we consider a start date and we set the following dates:

```
>> cfd=['01/01/2000'; '03/01/2000'; '03/15/2000'; '04/01/2000']
cfd =
01/01/2000
03/01/2000
03/15/2000
04/01/2000
```

Finally we use command xirr to obtain the annual IRR:

```
>> xirr(cf,cfd)
ans =
   2.0271
```

That is i*=202.71%

HOMEWORKS

19. A project requires an initial cash outlay of 2000 euro and is expected to generate 800 euro at the end of year 1 and 1600 at the end of year 2, at which time the project will terminate. Calculate the IRR of the project (analytically).

20. An investor pays 5000 euro today and he will receive 1200 euro at the end of each year for 5 year. Calculate the IRR of his investment (use MatLab).

21. Determine the annual IRR for the following project: {(-100,30,80);(0,2,4)} both analytically and by using MatLab then compare the results (the results can be slightly different since we use commercial year while MatLab uses actual calendar).

22. Determine the annual IRR for the following project using MatLab: {(-100,130,40,-50);(0,2,3,6)}.

[You can use MatLab for calculations]

SPOT INTEREST RATES AND FORWARD INTEREST RATES

ASSUMPTIONS

We now allow the rate of interest to vary with the duration of the investment.

We consider the case where investments over different horizons earn different rates of interest, although the principle of compounding still applies.

We consider two notions of interest rates, namely, the **spot rate of interest** and the **forward rate of interest**.

THE PROBLEM: Consider an investment at time 0 earning interest over t periods. We assume that the period of investment is fixed at the time of investment, but the rate of interest earned per period varies according to the investment horizon.

DEF. SPOT RATE OF INTEREST: we define i(0,t) as the **spot rate of interest**, which is the annualized effective rate of interest (IRR) for the period from time 0 to time t.

NOTICE THAT: To find the spot rate of interest in the asset market we consider **zero coupon bonds**: if we know their prices at time 0 (normally today) and the rembursment price (nominal value) at time t (the deadline), **the spot interest rate can be found simply requiring that the actual price is equal to the present value of the rembursement value.**

EX26: consider a zero coupon bond (ZCB) that is quoted 94.3, ending in 6 months, nominal value (rembursmente value) 100.

The spot rate of interest is i(0,0.5) is the annual interest rate associated to the investment in the ZCB from today to 6 months (0.5 years).

To compute i(0,0.5) we require the following equation to be realized

$$100 = 94.3\left(1 + i\left(0, \frac{1}{2}\right)\right)^{\frac{1}{2}} \quad \Rightarrow \quad i\left(0, \frac{1}{2}\right) = \left(\frac{100}{94.3}\right)^2 - 1 = 0.1245\,.$$

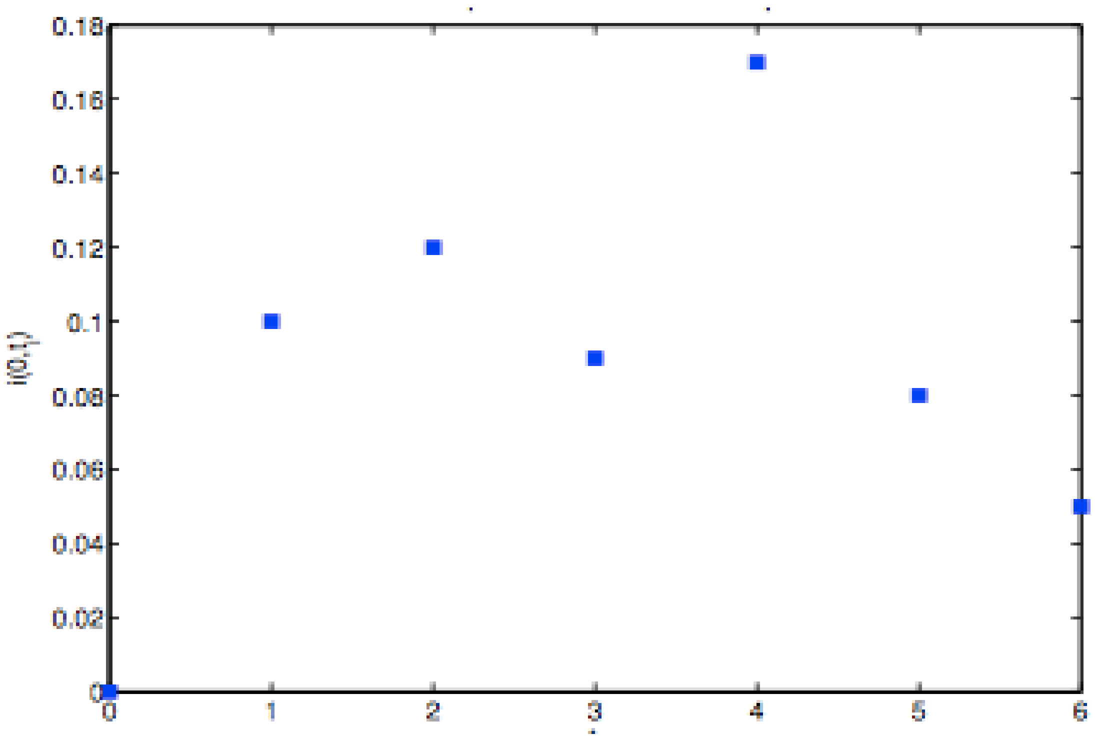
so that i(0,0.5)=12.45%.

DEF term structure of spot interest rates: A plot of i(0,t) against t is called the **yield curve**, and the mathematical relationship between i(0,t) and t is called the **term structure of the spot interest rates**.

Empirically the term structure can take **various shapes**:

flat term structure, downward sloping term structure, upward sloping term structure, not monotonic term structure.

EX27: let i(0,0)=0 , i(0,1)=0.1, i(0,2)=0.12, i(0,3)=0.9 i(0,4)=0.17 i(0,5)=0.08 i(0,6)=0.05. Then the term structure of spot interest rates is depicted in the following figure:



DEF. Forward interest rate: we define the (one-period) **forward rate of interest** as the interest rate that is determined today (time t=0) for an investment in ZCB from time t-1 to time t. We denote the forward rate of interest as i(0,t-1,t).

EX28: today we sign a contract to buy a ZCB at time t=2 (years) at price 98 whose deadline is time t=3 (nominal rembursment value is 100).

To find i(0,2,3) we have to solve the following equation:
100=98(1+i(0,2,3)) thus obtaining i(0,2,3)=0.0204.

Relation between spot and forward rates of interest.

The spot and forward rates are not free to vary independently of each other.

Assume that **the market is perfectly competitive** and that no arbitrage oportunities exist (i.e. it is not possible to gain profits without risk). Consider an horizon of t periods.

First strategy: single investment. Consider an investment in ZCB for the period (0,t). If an investor invests a unit amount at time 0 over t periods, the investment will accumulate to

$$(1 + i(0,t))^t$$

at time t.

Second strategy: rollover. Consider and investment of one unit payment at time 0 to period t-1, and enters into a forward agreement to invest the amount that will be disposable at time t-1, that is $(1+i(0,t-1))^{(t-1)}$, at time t-1 to earn the forward rate of i(0,t-1,t) for 1 period, thus obtaining the final amount

$$(1+i(0,t-1))^{(t-1)}(1+i(0,t-1,t))$$

at time t

As the market is perfectly competitive **the two strategies will give the same amount at time t** so that the following equation must hold:

$$(1 + i(0,t))^{t} = (1+i(0,t-1))^{(t-1)}(1+i(0,t-1,t)) \quad (***).$$

From (***) the forward rates can be obtained form the spot ones simply applying the following formula:

$$\mathbf{i(0,t-1,t)=(1 + i(0,t))^{t} /(1+i(0,t-1))^{(t-1)} - 1 \; (****)}$$

And the obtained forward rates are called the **implicit forward rates**.

NOTICE THAT: The quoted forward rates in the market may differ from the implicit forward rates in practice, as when the market is noncompetitive.

EX29: Suppose the spot rates of interest for investment horizons of 1, 2, 3 and 4 years are, respectively, 4%, 4.5%, 4.5%, and 5%. Calculate the forward rates of interest for t = 1, 2, 3 and 4.

We apply formula (****) as follows:

$$i(0,1,2) = \frac{(1+i(0,2))^2}{(1+i(0,1))} - 1 = \frac{(1.045)^2}{1.04} - 1 = 5.0024\%$$

$$i(0,2,3) = \frac{(1+i(0,3))^3}{(1+i(0,2))^2} - 1 = \frac{(1.045)^3}{(1.045)^2} - 1 = 4.5\%$$

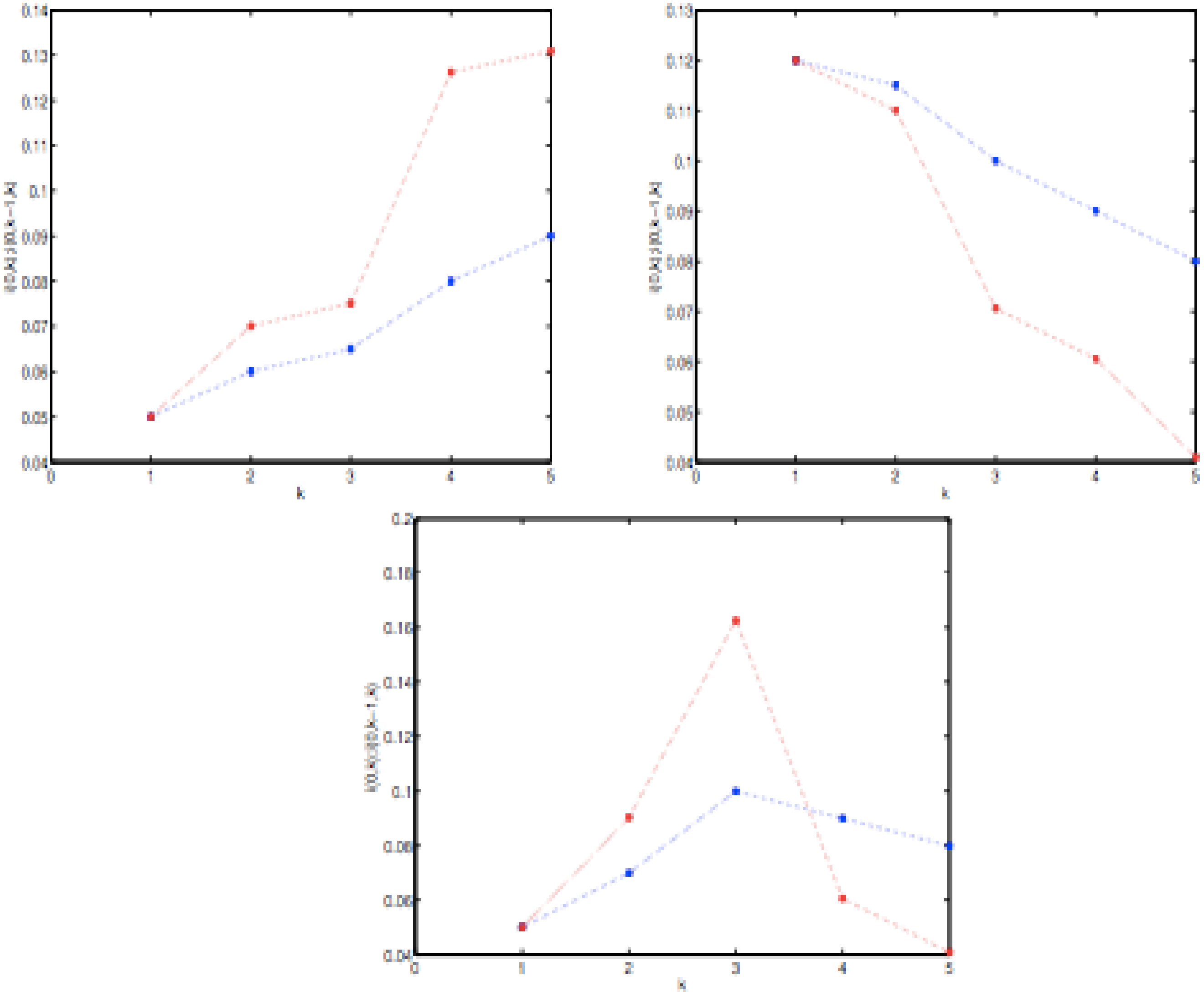
$$i(0,3,4) = \frac{(1+i(0,4))^4}{(1+i(0,3))^3} - 1 = \frac{(1.05)^4}{(1.045)^3} - 1 = 6.5144\%$$

DEF. Term structure of forward rates: The **term structure of the forward rates** is obtained while posing in the plane the uni-period forward interest rates i(0,t-1,t) related to each time t.

From formula (***) the following relationship between the term structure of forward and spot rates can be easlily obtained.

Dominance relationship: **if the yeld curve is upward (dawnward) sloping then the forward rate exceeds or dominate (is dominated by) the spot rate. When the spot structure is not monotonic then the spot and forward structure intersect each other.**

In the following picture the spot structure is in blue while the forward structure is in red.



DEF: Notice that formula (***) can be generalized to obtain **multi-period forward rates** given by i(0,t1,t2) that are rates fixed today for an investent starting at time t1 and ending at time t2:

$$(1+i(0,t2))^{t2}=(1+i(0,t1))^{t1}(1+i(0,t1,t2))^{(t2-t1)}$$

So that the related **multi-period forward rate** is given by:

$$i(0,t1,t2) = \left[\frac{(1+i(0,t2))^{t2}}{(1+i(0,t1))^{t1}}\right]^{1/t2-t1} - 1$$

EX30: Let be given $i(0,1)=0.075$, $i(0,2,3)=0.035$, $i(0,3)=0.05$.

The term structure of the spot rates of interest can be obtained.

We need to know $i(0,2)$. From $i(0,3)$ and $i(0,2,3)$ the following relation can be posed:
$(1+i(0,3))^3=(1+i(0,2))^2(1+i(0,2,3))$ so that $i(0,2)=((1+i(0,3))^3/(1+i(0,2,3)))^{0.5}-1$
Hence

$$i(0,2)=[1.05^3/1.035]^{0.5}-1=0.057$$

And also the term structure of the forward rates (uni-periodal) can be constructed as seen before. While the multi periodal forward rate $i(0,1,3)$ is obtained by posing:

$$(1+i(0,3))^3=(1+i(0,1))(1+i(0,1,3))^2$$

hence we have to compute $i(0,1,3)=[(1+i(0,3))^3/(1+i(0,1))]^{1/2}-1=$

$$=[1.05^3/1.075]^{0.5}-1=0.0377$$

Notice that: to calculate the present value of an annuity given the term structure it is sufficient to use the spot rate of interest related to the timing of each payment.

EX31: let i(0,0.5)=0.04, i(0,1)=0.045 and i(0,1.5)=0.05, and recall that these are annual interest rates. Consider the following payments: 200 euro in 6 months, 300 euro in one year, 250 euro in 3 semesters. Then the present value of the cash flow is given by:

$$V=200(1.04)^{-0.5}+300(1.045)^{-1}+250(1.05)^{-1.5}=715.56$$

HOMEWORKS

23. Consider the following three ZCB: (a) Actual price 96 and deadline 6 months, (b) Actual price 95 and deadline 1 year, (c) Actual price 93 and deadline 2 years. Determine the related spot rates of interests.

24. Let be given the following spot and forward rates of interests: i(0,1)=0.05, i(0,2)=0.07 and i(0,2,3)=0.06. Determine i(0,1,2) and i(0,3). Also depict the spot and forward term structures.

25. Let i(0,2)=0.075, i(0,1,3)=0.08, i(0,3)=0.065, i(0,4)=0.08 determine all the spot and forward rates (both uni-period and multi-period) that can be determined under the no arbitrage assumption.

26. Let i(0,1/12)=0.03, i(0,2/12)=0.04 and i(0,1/12,3/12)=0.05. Determine the structure of the spot rates and the present value of the following cash flow: 10 euro after 1 months, 10 euro after 2 months, 110 euro after 3 months.

[You can use MatLab for calculations]

DURATION

PROBLEM: we consider a bond or in general a cash flow producing payments at different times, for instance:

$$OF=\{(x_1,x_2,x_3,\ldots,x_n);(t_1,t_2,t_3,\ldots,t_n)\}$$

$x_k \geq 0$ then we want to find a way to summarize the **horizon** of this cash flow.

To the scope **we have to consider what follows**:

-the payments are due at times t_1, t_2,…., t_n so that our measure must consider the final deadline (t_n) and also the intermediates deadlines (from t_1 to t_{n-1})

-the amounts of the payments due at any time (in order to give more importance to the deadlines associated to payments of greater amount)

-the fact that all the amounts must be comparable so that they must be evaluated at the same period (for instance by calculating the present value)

-when computing the present value of each payment: (1) a fixed interest rate can be considered (i.e. a constant yeld curve is assumed) or (2) we can consider the spot interest rates referred to each timing.

DEF: The **DURATION** is the weighted average of the time of the cash flows (t_1, t_2, ..., t_n) and wheights are given by the present values of the cash flows (not their nominal values) by **taking into account the term structure of spot rates of interests.**

$$D = \frac{t_1 x_1 (1+i(0,t_1))^{-t_1} + t_2 x_2 (1+i(0,t_2))^{-t_2} + \ldots + t_n x_n (1+i(0,t_n))^{-t_n}}{x_1 (1+i(0,t_1))^{-t_1} + x_2 (1+i(0,t_2))^{-t_2} + \ldots + x_n (1+i(0,t_n))^{-t_n}}$$

or similarly

$$D = t_1 p_1 + t_2 p_2 + \ldots + t_n p_n$$

where:

$$p_k = \frac{x_k (1+i(0,t_k))^{-t_k}}{x_1 (1+i(0,t_1))^{-t_1} + x_2 (1+i(0,t_2))^{-t_2} + \ldots + x_n (1+i(0,t_n))^{-t_n}},$$

$$k = 1, 2, \ldots, n.$$

NOTICE THAT: since D is an average value, then D is between t_1 and t_n

EX32: Consider the following cash flow

$$OF1 = \{(1, 1, 1, 100); (1, 2, 3, 4)\}$$

and the following term structure of spot interest rates

$$i(0,1) = 0.07,\ i(0,2) = 0.075,\ i(0,3) = 0.065,\ i(0,4) = 0.08.$$

In order to calculate the duration, first we calculate the present value of each payment of the cash flow OF1:

$$\begin{aligned} V_1 &= 1(1.07)^{-1} = 0.9346 \\ V_2 &= 1(1.075)^{-2} = 0.8653 \\ V_3 &= 1(1.065)^{-3} = 0.8278 \\ V_4 &= 100(1.08)^{-4} = 73.5030 \end{aligned}$$

So that the present value of the cash flow is given by

$$V = V_1 + V_2 + V_3 + V_4 = 76.1307\,.$$

Finally the duration is obtained:

$$D_1 = \frac{1 \cdot 0.9346 + 2 \cdot 0.8653 + 3 \cdot 0.8278 + 4 \cdot 73.5030}{76.1307} = 3.9296.$$

DEF: If the duration is computed by using a constant interest rate, that is by assuming **$i(0,t_k)=i$ for all k**, then the obtaining duration is called **MACAULY DURATION.**

EX33: Calculate the Macaulay duration of a 4-year annual coupon bond with 6% coupon and a yield to maturity of 5.5%.

The cash flow is as follows: OF={(1,2,3,4);(6,6,6,106)}, we compute the present value of each amount of the cash flow (C_i) by using i=0.055 and obtain the following:

t	C_t	$\mathrm{PV}(C_t)$
1	6	5.6872
2	6	5.3907
3	6	5.1097
4	106	85.5650
Total		101.7526

The price of the bond P is equal to the sum of 101.7526.
Finally the **Macauly duration** is given by:

$$\frac{1(5.6872) + 2(5.3907) + 3(5.1097) + 4(85.5650)}{101.7526} = 3.6761$$

SENSITIVITY

While the **Macaulay duration** was originally proposed to measure the average horizon of an investment, it turns out that it can be used to measure the price sensitivity of the investment with respect to interest-rate changes.

We consider the price of a bond as given by the present value of the cash flow by using a constant interest rate as follows:

$$V_0(\underline{x}, i^*) = x_1(1+i^*)^{-1} + x_2(1+i^*)^{-2} + \ldots + x_n(1+i^*)^{-n}.$$

The **price sensitivity with respect to a change in the interest rate** can be measured by the derivative of the present value V with respect to i* related to the initial value V, that is V'/V.

From $V_0' = -x_1(1+i^*)^{-2} - 2x_2(1+i^*)^{-3} + \ldots - nx_n(1+i^*)^{-n-1} =$

$$-\frac{1}{1+i^*}\left[x_1(1+i^*)^{-1} + 2x_2(1+i^*)^{-2} + \ldots + nx_n(1+i^*)^{-n}\right].$$

We have that

$$\left[x_1(1+i^*)^{-1} + 2x_2(1+i^*)^{-2} + \ldots + nx_n(1+i^*)^{-n}\right] = D(i^*)V_0(\underline{x}, i^*)$$

Hence

$$V_0' = -\frac{1}{1+i^*}D(i^*)V_0(\underline{x}, i^*).$$

And finally

$$\frac{V_0'}{V_0(\underline{x}, i^*)} = -\frac{1}{1+i^*}D(i^*).$$

DEF: the quantity

$$-\frac{1}{1+i^*}D(i^*)$$

is called **MODIFIED DURATION** and it measures the percentage decrease of the value of the investment per unit increase in the rate of interest.

Notice that: since the modified duration is negative, then as long as the interest rate increases then the bond price decreases, furthermore bonds with higher duration are more sensitive to variations in the interest rate.

EX34: from EX31 we can now calculate the modified duration: it is given by MD=-3.6761/1.055 = -3.4845.
Thus, the bond drops in value by 3.4845% per 1 percentage point increase (not percentage increase) in interest rate.

Notice that: as the bond price and interest rate **relationship is nonlinear**, the modified duration (MD) can measure the price variation w.r.t. interest rate only approximately, that is it applies to infinitesimal variations of the current rate of interest (i.e. in an opportune neighborhood of the initial rate).

HOMEWORKS

27. Consider the following cash flows: OF1={(1,2,3,4);(5,5,5,105)} and OF2={(1,2,3,4,5);(21,21,21,21,21)}. Consider also the following spot interest rates: i(0,1)=0.06, i(0,2)=0.062, i(0,3)=0.065, i(0,4)=0.07 and i(0,5)=0.075. Calculate the durations.

28. Calculate the Macaulay duration of a 2-year semiannual coupon bond with 4% coupon per semester and a yield to maturity of 4.8% annual. Calculate also the modified duration.

[You can use MatLab for calculations]